SUPPORT FUNCTIONS AND MEAN WIDTH FOR α-CONCAVE FUNCTIONS

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ABSTRACT. In this paper we extend some notions, previously defined for log-concave functions, to the larger domain of so-called α-concave functions. We begin with a detailed discussion of support functions – first for log-concave functions, and then for general α-concave functions. We continue by defining mean width, and proving some basic results such as an Urysohn type inequality. Finally, we demonstrate how such geometric results can imply Poincaré type inequalities.

1. SUPPORT FUNCTIONS AND α-CONCAVE FUNCTIONS

A well known construction in classic convexity is the support function of a convex body. Let $\emptyset \neq K \subseteq \mathbb{R}^n$ be a closed, convex set. Then its support function is a function $h_K : \mathbb{R}^n \to (-\infty, \infty]$, defined by

$$h_K(y) = \sup_{x \in K} \langle x, y \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean structure on $\mathbb{R}^n$.

To begin our discussion, we will briefly state a few basic properties of support functions. A more detailed account, together with all of the relevant proofs, can be found for example in section 1.7 of [12]. Define

$$\mathcal{K}^n = \{\emptyset \neq K \subseteq \mathbb{R}^n : K \text{ is closed and convex}\}.$$

For every $K \in \mathcal{K}^n$ its support function h_K belongs to

$$\mathcal{H}^n = \left\{ \varphi : \mathbb{R}^n \to (-\infty, \infty] : \begin{array}{l} \varphi \text{ is convex, lower semicontinuous and} \\ \text{positively homogenous with } \varphi(0) = 0 \end{array} \right\}.$$

Furthermore, the map $\mathcal{T} : \mathcal{K}^n \to \mathcal{H}^n$ sending K to h_K has the following properties:

(P1) $\mathcal{T}$ is bijective.
(P2) $\mathcal{T}$ is order preserving: $K_1 \subseteq K_2$ if and only if $\mathcal{T}K_1 \leq \mathcal{T}K_2$ (here and after, $f \leq g$ means $f(x) \leq g(x)$ for all x)
(P3) $\mathcal{T}$ is additive: For every K_1, K_2 we have $\mathcal{T}(K_1 + K_2) = \mathcal{T}K_1 + \mathcal{T}K_2$, where the addition on the left hand side is Minkowski addition:

$$K_1 + K_2 = \{x + y : x \in K_1 \text{ and } y \in K_2\}.$$

It turns out that these properties suffice to characterize $\mathcal{T}$ uniquely, up to a linear change of variables. In fact, one can do even better: From the work of Gruber in [9] it is easy to deduce the following:

Theorem. *Assume $\mathcal{T} : \mathcal{K}^n \to \mathcal{H}^n$ satisfies (P1) and (P2) for $n \geq 2$. Then there exists an invertible affine map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{T}(K) = h_{B(K)}$ for all $K \in \mathcal{K}^n$.*

Similarly, in [2] Artstein-Avidan and Milman prove:

Theorem. *Assume $\mathcal{T} : \mathcal{K}^n \to \mathcal{H}^n$ satisfies (P1) and (P3) for $n \geq 1$. Then there exists an invertible linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{T}(K) = h_{B(K)}$ for all $K \in \mathcal{K}^n$.*

We will now shift our attention from bodies to functions. In recent years, many notions and results from convexity were generalized from the class of convex bodies to larger domains. One usual choice for such a domain is the class of *log-concave* functions. To give an exact definition, we can define

$$\mathrm{Cvx}(\mathbb{R}^n) = \left\{ \varphi : \mathbb{R}^n \to (-\infty, \infty] : \begin{array}{c} \varphi \text{ is convex, lower semicontinuous and} \\ \varphi(x) < \infty \text{ for some } x \end{array} \right\},$$

and then the class of log-concave functions is simply

$$\mathrm{LC}(\mathbb{R}^n) = \left\{ e^{-\varphi} : \varphi \in \mathrm{Cvx}(\mathbb{R}^n) \right\}.$$

Put differently, a function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if $(-\log f)$ is a convex function and if f is upper semicontinuous and not identically zero. There is a natural embedding of $\mathcal{K}^n$ into $\mathrm{LC}(\mathbb{R}^n)$, which maps every convex body K to its characteristic function

$$\mathbf{1}_K = \begin{cases} 1 & x \in K \\ 0 & \text{otherwise.} \end{cases}$$

Some notions from convexity extend easily to the class of log-concave functions. For example, since

$$\int_{\mathbb{R}^n} \mathbf{1}_K(x) dx = \mathrm{Vol}(K),$$

one can say that the integral of a function $\int f$ extends the notion of the volume of a convex body $\mathrm{Vol}(K)$. Other extensions might not be as obvious. It is by now standard to extend the notion of Minkowski addition by the operation known as sup-convolution, or Asplund sum: For $f, g \in \mathrm{LC}(\mathbb{R}^n)$ we define their sum by

$$(f \star g)(x) = \sup_{y+z=x} f(y)g(z).$$

Notice that this is indeed a generalization, in the sense that $\mathbf{1}_K \star \mathbf{1}_T = \mathbf{1}_{K+T}$.

Similarly, it is standard to extend the notion of support function by defining

$$h_f = (-\log f)^*$$

for $f \in \mathrm{LC}(\mathbb{R}^n)$. The $*$ here denotes the classic Legendre transform, defined by

$$\varphi^*(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - \varphi(x)).$$

Again, this is a proper generalization, in the sense that $h_{\mathbf{1}_K} = h_K$, as one easily checks.

It turns out that support functions of log-concave functions share most of the important properties of support functions of convex bodies. More specifically, the function $\mathcal{T} : \mathrm{LC}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ mapping f to h_f has the following properties:

(Q1) $\mathcal{T}$ is bijective.
(Q2) $\mathcal{T}$ is order preserving: $f_1 \leq f_2$ if and only if $\mathcal{T}f_1 \leq \mathcal{T}f_2$.
(Q3) $\mathcal{T}$ is additive: For every f_1, f_2 we have $\mathcal{T}(f_1 \star f_2) = \mathcal{T}f_1 + \mathcal{T}f_2$.

Again, one can use properties (Q1)-(Q3) to uniquely characterize the support function up to a linear change of variables. In [1] Artstein-Avidan and Milman prove

Theorem. *Assume $\mathcal{T} : LC(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ satisfies (Q1) and (Q2). Then there exists an invertible affine map $B : \mathbb{R}^n \to \mathbb{R}^n$, constants C_1, C_2 and a vector $v \in \mathbb{R}^n$ such that*

$$(\mathcal{T}f)(x) = C_1 \cdot h_f(Bx) + \langle x, v \rangle + C_2$$

for all $f \in LC(\mathbb{R}^n)$.

Additionally, in [2] the same authors prove the following:

Theorem. *Assume $\mathcal{T} : LC(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ satisfies (Q1) and (Q3). Then there exists an invertible affine map $B : \mathbb{R}^n \to \mathbb{R}^n$ and a constant C such that*

$$(\mathcal{T}f)(x) = C \cdot h_f(Bx)$$

for all $f \in LC(\mathbb{R}^n)$.

The last two theorems actually serve an important purpose. The definition of the support function h_K of a convex body exists for a long time, and has proven itself to be extremely useful. The definition of the support function h_f of a log-concave function, however, is much newer, and it is reasonable to debate the question of whether this definition is the "right" one. These theorems give us a way to justify our definitions: If, for example, one believes that (Q1) and (Q2) are reasonable properties that any definition will have to satisfy, then the exact definition follows immediately.

However, as was pointed out by Vitali Milman, the assumption (Q1) may not be as innocent as it first appears. Injectivity of $\mathcal{T}$ is fairly natural, and follows easily from property (Q2) as well. Surjectivity, on the other hand, is a more delicate matter. After all, the support function h_K of a convex body is not an arbitrary convex function, but always a positively homogeneous one. It is possible that $\mathrm{Im}\mathcal{T}$ should also be a proper subset of $\mathrm{Cvx}(\mathbb{R}^n)$, and that in order to get every function $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ as a support function one should increase the domain $\mathrm{LC}(\mathbb{R}^n)$ even further.

As it turns out, there is a well known way to extend $\mathrm{LC}(\mathbb{R}^n)$ to a larger class of functions. Consider the following definition:

Definition 1. Fix $-\infty \leq \alpha \leq \infty$. We say that a function $f : \mathbb{R}^n \to [0, \infty)$ is α-concave if f is supported on some convex set Ω, and for every $x, y \in \Omega$ and $0 \leq \lambda \leq 1$ we have

$$f(\lambda x + (1-\lambda)y) \geq [\lambda f(x)^\alpha + (1-\lambda) f(y)^\alpha]^{\frac{1}{\alpha}}.$$

For $\alpha = -\infty, 0, \infty$ we understand this inequality in the limit sense. This means that f is $(-\infty)$-concave if

$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\},$$

is 0-concave if it is log-concave:

$$f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda},$$

and ∞-concave if

$$f(\lambda x + (1 - \lambda)y) \geq \max\{f(x), f(y)\},$$

which implies that f is constant on Ω.

In this definition we follow the conventions of Brascamp and Lieb in [6], but the notion can be traced to the works of Avriel [3] and Borell [5]. The interested reader may also consult [4] for applications more similar in spirit to this paper.

One of the goals of this paper is to demonstrate how some constructions, usually carried out for log-concave functions, can also be carried out for general α-concave functions. These constructions will include the support function, and the mean width. This discussion, together with a more systematic treatment of α-concave functions, will appear in section 3. For now, let us just mention the fact that if $\alpha_1 < \alpha_2$, then every α_2-concave function, is also α_1-concave. We can use this fact to generate the following example:

Example 2. Fix $-\infty < \alpha < 0$. The claim that $f : \mathbb{R}^n \to [0, \infty)$ is α-concave is equivalent to saying that f^α is convex. This means we can write every α-concave function, and hence every log-concave function, as

$$f(x) = [1 - \alpha \cdot \varphi(x)]^{\frac{1}{\alpha}}$$

for some convex function φ (notice that $-\alpha$ is positive). Define $\mathcal{T}_\alpha : \mathrm{LC}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ by

$$\mathcal{T}_\alpha f = \varphi^*.$$

It is easy to verify that $\mathcal{T}_\alpha$ extends the classic support function, in the sense that if $f = \mathbf{1}_K$ then $\mathcal{T}_\alpha f = h_K$. It is equally easy to check that this $\mathcal{T}_\alpha$ satisfies property (Q2), but not property (Q1). Since $\mathcal{T}_\alpha f$ is in general very different from h_f, we see that the Artstein-Milman characterization theorem fails completely without the surjectivity assumption.

To gain an insight into the origins of this example, notice that as $\alpha \to 0$ we have $\varphi \to (-\log f)$, so $\mathcal{T}_\alpha f \to h_f$, at least on a heuristic level. Intuitively, one may think of $\mathcal{T}_\alpha f$ as the "right" definition of the support function of an α-concave function, and the standard definition is just the special case $\alpha = 0$. We will revisit this point of view in section 3.

It is interesting to note that the above example does not satisfy property (Q3), so at least in this sense it is less natural then the standard construction. Vitali Milman asked whether it possible to assume *both* (Q2) and (Q3), and prove a characterization theorem which does not require surjectivity. Indeed, this is the case:

Theorem 3. *Assume we are given a function $\mathcal{S} : LC(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ and an operation $\oplus : LC(\mathbb{R}^n) \times LC(\mathbb{R}^n) \to LC(\mathbb{R}^n)$ with the following properties:*

(1) $\mathcal{S}$ is order preserving: $f \leq g$ if and only if $\mathcal{S}f \leq \mathcal{S}g$.
(2) $\mathcal{S}$ extends the usual support functional: If $f = \mathbf{1}_K$, then $\mathcal{S}f = h_K$.
(3) $\mathcal{S}(f \oplus g) = \mathcal{S}f + \mathcal{S}g$.

Then we must have

$$(\mathcal{S}f)(x) = \frac{1}{C}h_f(Cx)$$

for some $C > 0$, and $f \oplus g = f \star g$.

Note that even though we do not assume surjectivity a priori, it follows a posteriori that $\mathcal{S}$ must be onto $\mathrm{Cvx}(\mathbb{R}^n)$. Another interesting feature of the theorem is that our third assumption is somewhat weaker than (Q3), as we only need to assume that S is additive with respect to *some* addition operation $\oplus$. Therefore this theorem characterizes not only the support function, but the sup-convolution operation as well.

Theorem 3 will follow easily from the following theorem:

Theorem 4. *Assume a function $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ satisfies the following:*

(1) *$\mathcal{T}$ is order preserving: $\varphi \leq \psi$ if and only if $\mathcal{T}\varphi \leq \mathcal{T}\psi$.*
(2) *If φ is a positively homogeneous function then $\mathcal{T}\varphi = \varphi$.*
(3) *The set $Im\mathcal{T} = \{\mathcal{T}\varphi : \varphi \in Cvx(\mathbb{R}^n)\}$ is closed under pointwise addition.*

Then $(\mathcal{T}\varphi)(x) = \frac{1}{C}\varphi(Cx)$ for some $C > 0$.

Proof of the reduction. Assume Theorem 4 holds. In order to prove Theorem 3, choose $\mathcal{S} : \mathrm{LC}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ satisfying all of the assumptions. Define $\mathcal{T} : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ by

$$\mathcal{T}\varphi = \mathcal{S}\left(e^{-\varphi^*}\right).$$

It it easy to see that $\mathcal{T}$ satisfies all of the assumptions of theorem 4, so

$$\left(\mathcal{S}\left(e^{-\varphi^*}\right)\right)(x) = (\mathcal{T}\varphi)(x) = \frac{1}{C}\varphi(Cx)$$

for some $C > 0$. For every $f \in \mathrm{LC}(\mathbb{R}^n)$ define $\varphi = h_f$, and notice that $e^{-\varphi^*} = f$. Hence we get

$$(Sf)(x) = \frac{1}{C}h_f(Cx)$$

like we wanted. In particular, for every $f, g \in \mathrm{LC}(\mathbb{R}^n)$ we will get

$$S(f \star g) = Sf + Sg = S(f \oplus g),$$

and since $\mathcal{S}$ is injective, $f \oplus g = f \star g$. This completes the proof. $\qquad\square$

The rest of this paper is organized as follows: Section 2 will include the proof of theorem 4. The proof is rather long, and is composed of several independent ingredients. Some of the ingredients are fairly standard by now, but some are probably new. Let me thank Alexander Segal and Boaz Slomka for providing some of these arguments. Section 3 will be devoted to α-concave functions. We will extend the notions discussed in this section, specifically addition and support functions, to the realm of α-concave functions. Finally, we will define the mean width of an α-concave function, and generalize some known results to the new setting.

2. Proof of theorem 4

We will now prove Theorem 4. As the proof is quite long, we will divide it into several parts:

1. "Negatively affine" functions. Fix a vector $a \in \mathbb{R}^n$. The linear function $\ell(x) = \langle x, a \rangle$ is 1-homogeneous, so $\mathcal{T}\ell = \ell$. We will now deal with affine functions of the form

$$\ell_1(x) = \langle x, a \rangle + c$$

for some $c < 0$ (we call such functions "negatively affine functions"). Since $\ell_1 \leq \ell$ and $\mathcal{T}$ is order preserving we must have $\mathcal{T}\ell_1 \leq \mathcal{T}\ell = \ell$, and it follows immediately that

$$\left(\mathcal{T}\ell_1\right)(x) = \langle x, a \rangle + c'$$

for some constant $c' < 0$.

Later we will show that $\mathrm{Im}\mathcal{T}$ contains all affine functions. For now, just notice that if $n \in \mathbb{N}$ then we can write

$$\langle x, a \rangle + nc' = n\left[\langle x, a \rangle + c'\right] + \langle x, a - na \rangle,$$

and since $\mathrm{Im}\mathcal{T}$ is closed under addition we get that $\langle x, a \rangle + nc' \in \mathrm{Im}\mathcal{T}$. In particular, there exists a sequence $c_n \to -\infty$ such that $\langle x, a \rangle + c_n \in \mathrm{Im}\mathcal{T}$.

Finally, notice that if a negatively affine function

$$\ell'(x) = \langle x, a \rangle + c'$$

is in $\mathrm{Im}\mathcal{T}$, then we can apply the same reasoning as above for $\mathcal{T}^{-1}$ and conclude that $\left(\mathcal{T}^{-1}\ell'\right)(x) = \langle x, a \rangle + c$ for some $c < 0$.

2. "Positively affine" functions. Assume now that we are given a function $\ell(x) = \langle x, a \rangle + c$ with $a \in \mathbb{R}^n$ and $c > 0$. We want to prove that $\varphi' = \mathcal{T}\ell$ is also affine. To do so we will need the notion of a tangent. Remember that an affine function ℓ is tangent to a function $\psi \in \mathrm{Cvx}(\mathbb{R}^n)$ if $\ell \leq \psi$, but $\ell + \varepsilon \not\leq \psi$ for every $\varepsilon > 0$. It is well known that

$$\psi = \sup\left\{\ell : \ell \text{ is tangent to } \psi\right\}.$$

Let

$$\ell_1'(x) = \langle x, b \rangle + d_1'$$

be any tangent to φ'. As we saw before, we can choose a constant $d_2' \leq \min\left(d_1', 0\right)$ such that

$$\ell_2'(x) = \langle x, b \rangle + d_2' \in \mathrm{Im}\mathcal{T}.$$

Since $\ell_2' \leq \ell_1' \leq \varphi' = \mathcal{T}\ell$, it follows that $\ell_2 \leq \ell$, where

$$\ell_2(x) = \left(\mathcal{T}^{-1}\ell_2'\right)(x) = \langle x, b \rangle + d_2,$$

and then we must have $b = a$. In other words, every tangent to φ' is of the form $\langle x, a \rangle + d$ for some d, and this can only happen if

$$\varphi'(x) = \langle x, a \rangle + c'$$

for some c'. Since $\ell(x) \geq \langle x, a \rangle$ we of course have $c' > 0$.

Again, if $\ell'(x) = \langle x, a \rangle + c'$ for $c' > 0$ happens to be in $\text{Im}\mathcal{T}$, we can repeat the argument in reverse and conclude that

$$\left(\mathcal{T}^{-1}\ell'\right) = \langle x, a \rangle + c$$

for some $c > 0$.

3. Surjectivity on affine functions. So far we have seen that the image of any affine function is affine, with parallel graph. We will now show that all affine functions are in $\text{Im}\mathcal{T}$. Fix $a \in \mathbb{R}^n$, and define $f_a : \mathbb{R} \to \mathbb{R}$ according to the formula

$$\mathcal{T}\left(\langle x, a \rangle + c\right) = \langle x, a \rangle + f_a(c).$$

Also define

$$H_a = \text{Im}(f_a) = \{c' \in \mathbb{R} : \ \langle x, a \rangle + c' \in \text{Im}\mathcal{T}\}.$$

Notice that H_a is closed under addition: If $c_1, c_2 \in H_a$ then

$$\langle x, a \rangle + (c_1 + c_2) = [\langle x, a \rangle + c_1] + [\langle x, a \rangle + c_2] + \langle x, -a \rangle \in \text{Im}\left(\mathcal{T}\right),$$

so $c_1 + c_2 \in H_a$ as well. We will now apply the following Lemma:

Lemma 5. *Assume $H \subseteq \mathbb{R}$ is a subset with the following properties:*

    (1) *If $x, y \in H$ then $x + y \in H$.*
    (2) *There exists $x \in H$ such that $x > 0$.*
    (3) *There exists $x \in H$ such that $x < 0$.*

Then H is either a cyclic subgroup of $\mathbb{R}$ or dense in $\mathbb{R}$.

This result, or slight variations thereof, is well known in some fields. For the sake of completeness, we will prove Lemma 5 after we finish proving Theorem 4.

By the above discussion we see that H_a satisfies the hypotheses of Lemma 5, so it is either cyclic or dense. Since f_a is injective, H_a must be uncountable, and since cyclic groups are all countable, H_a must be dense. Now $f_a : \mathbb{R} \to \mathbb{R}$ is a monotone function with dense image, and it is an easy exercise that all such functions are onto. Therefore $H_a = \mathbb{R}$ and we proved the desired result.

4. Delta functions. We will return to affine functions shortly, but before we do we need to discuss delta functions. For $a \in \mathbb{R}^n$ and $c \in \mathbb{R}$ define

$$\delta_{a,c}(x) = \begin{cases} c & x = a \\ \infty & \text{otherwise.} \end{cases}$$

Our goal is to show that delta functions are mapped to delta functions. Assume by contradiction that $\varphi' = \mathcal{T}\delta_{a,c}$ is not a delta function, so there exists $b_1 \neq b_2$ such that $\varphi'(b_1), \varphi'(b_2) < \infty$. We will divide the proof into two cases:

Case 1. There exists a constant $\lambda > 0$ such that $\lambda b_1 = b_2$. Without loss of generality we can assume $\lambda > 1$ (otherwise, swap b_1 and b_2 in the following proof). Define two functions

$$\begin{aligned} \psi_1'(x) &= |x| \\ \psi_2'(x) &= \frac{1 + \lambda}{2}|b_1| \end{aligned}$$

(where $|\cdot|$ denotes the Euclidean norm). $\psi_1' \in \mathrm{Im}\mathcal{T}$ as a positively homogeneous function, and $\psi_2' \in \mathrm{Im}\mathcal{T}$ as a constant, hence affine, function. Therefore if we define $\rho_i' = \varphi' + \psi_i'$ for $i = 1, 2$ then $\rho_i' \in \mathrm{Im}\mathcal{T}$ as well. Define $\rho_i = \mathcal{T}^{-1}(\rho_i')$. Since $\rho_i' \geq \varphi' = \mathcal{T}\delta_{a,c}$, we have $\rho_i \geq \delta_{a,c}$, so $\rho_i = \delta_{a,c_i}$ for some c_i. In particular, we must have $\rho_1 \geq \rho_2$ or vice versa. But this implies that ρ_1', ρ_2' are comparable as well, which is a contradiction:

$$\rho_1'(b_1) = \varphi'(b_1) + |b_1| \quad < \quad \varphi'(b_1) + \frac{1+\lambda}{2}|b_1| = \rho_2'(b_1)$$

$$\rho_1'(b_2) = \varphi'(b_2) + \lambda|b_1| \quad > \quad \varphi'(b_2) + \frac{1+\lambda}{2}|b_1| = \rho_2'(b_2).$$

Case 2. Now assume b_1 and b_2 are not on the same ray. In this case define

$$\psi_1'(x) \quad = \quad \begin{cases} 0 & x \in \mathbb{R}^+ b_1 \\ \infty & \text{otherwise} \end{cases}$$

$$\psi_2'(x) \quad = \quad 1,$$

and the rest of the proof is the same as in the previous case.

In both cases we arrived at a contradiction, so we proved that indeed φ' is a delta function like we wanted.

5. Tangents. Our next simple claim is that an affine function ℓ is tangent to $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ if and only if $\ell' = \mathcal{T}\ell$ is tangent to $\varphi' = \mathcal{T}\varphi$. Indeed, if ℓ is tangent to φ, then we immediately get $\ell' \leq \varphi'$. If $\ell' + \varepsilon \leq \varphi'$ for some $\varepsilon > 0$ then $\mathcal{T}^{-1}(\ell' + \varepsilon)$ is an affine function such that

$$\ell < \mathcal{T}^{-1}(\ell' + \varepsilon) \leq \varphi,$$

which is impossible. The other direction is proven in exactly the same way.

In particular, if $\varphi = \delta_{a,c}$, then ℓ is tangent to φ if and only if $\ell(a) = c$. Therefore if ℓ is an affine function and $a \in \mathbb{R}^n$, we can always find $b \in \mathbb{R}^n$ such that

$$\mathcal{T}\left(\delta_{a,\ell(a)}\right) = \delta_{b,(\mathcal{T}\ell)(b)}.$$

6. Collinearity. We will identify every affine map $\ell(x) = \langle x, a \rangle + c$ with the point $p_\ell = (a, c) \in \mathbb{R}^{n+1}$. Under this identification our map $\mathcal{T}$ induces a bijection $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by

$$F(p_\ell) = p_{\mathcal{T}\ell}.$$

Our current goal is to show that F^{-1} maps collinear points into collinear points. The following lemma will prove useful:

Lemma 6. *Assume ℓ_1, ℓ_2, ℓ_3 are 3 affine functions such that ℓ_1 and ℓ_2 are not parallel. Then $p_{\ell_1}, p_{\ell_2}, p_{\ell_3}$ are collinear if and only if whenever $\ell_1(x_0) = \ell_2(x_0)$ we also have $\ell_1(x_0) = \ell_3(x_0)$.*

Again, we will postpone the proof of Lemma 6 until the end of the proof of Theorem 4. For now we will use this lemma to prove the result: Assume $\ell_1', \ell_2', \ell_3'$ are collinear affine functions, and define $\ell_i = \mathcal{T}^{-1}\ell_i'$. If ℓ_1' and ℓ_2' are parallel then all six functions are parallel to each other and there is nothing to prove.

In the general case, assume a is any point such that

$$\ell_1(a) = \ell_2(a) = c,$$

and define $\varphi = \delta_{a,c}$. We've already seen that in this case we must have

$$\mathcal{T}\varphi = \delta_{b,d},$$

where $\ell_1'(b) = d = \ell_2'(b)$. Since $\ell_1', \ell_2', \ell_3'$ are collinear we get from Lemma 6 that $\ell_3'(b) = d$ as well. This implies that ℓ_3' is tangent to $\delta_{b,d}$, so ℓ_3 is tangent to $\delta_{a,c}$ and $\ell_3(a) = c$. Again by Lemma 6 we get that ℓ_1, ℓ_2, ℓ_3 are collinear like we wanted.

By the fundamental theorem of affine geometry, it now follows that F^{-1} is affine, so F is affine as well (for an exact formulation of the fundamental theorem and a sketch of the proof, the reader may consult [1]). This means that we can write

$$\mathcal{T}\left(\langle x, a\rangle + c\right) = \langle x, a\rangle + \langle a, v\rangle + \gamma c$$

for some constants $\gamma \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ (which are of course independent of a and c).

We know that $\mathcal{T}\left(\langle x, a\rangle\right) = \langle x, a\rangle$, so $\langle a, v\rangle = 0$ for all $a \in \mathbb{R}^n$, which implies $v = 0$. Also, for $\mathcal{T}$ to be order preserving, we must have $\gamma > 0$.

7. Finishing the proof. We now know that

$$\mathcal{T}\left(\langle x, a\rangle + c\right) = \langle x, a\rangle + \gamma c$$

for some $\gamma > 0$. Remember that in the statement of Theorem 4 we had one degree of freedom - we don't want to prove that $\mathcal{T}\varphi = \varphi$, but that $(\mathcal{T}\varphi)(x) = \frac{1}{C}\varphi(Cx)$ for some $C > 0$. We will now use this degree of freedom and assume that $\gamma = 1$ (formally, this means we replace $\mathcal{T}$ with $\widetilde{\mathcal{T}}$ defined by $\left(\widetilde{\mathcal{T}}\varphi\right)(x) = \frac{1}{\gamma}(\mathcal{T}\varphi)(\gamma x)$. We will keep using the notation $\mathcal{T}$ for the new function).

For every function $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ and any affine ℓ, we know that ℓ is tangent to φ if and only if $\mathcal{T}\ell = \ell$ is tangent to $\mathcal{T}\varphi$. In other words, φ and $\mathcal{T}\varphi$ have exactly the same tangents, so $\mathcal{T}\varphi = \varphi$ and our proof is finally complete.

8. Proofs of the lemmas.

Proof of Lemma 5. First we note that it is enough to prove that H is either cyclic or dense in $\mathbb{R}^+ = [0, \infty)$. Indeed, assume that H is dense in $\mathbb{R}^+$. We know that there exists an element $x < 0$ in H. If $y \in \mathbb{R}$ is any number, we can choose $n \in \mathbb{N}$ so large that $y - nx > 0$. Now we can find a sequence $\{x_k\} \subseteq H$ such that $x_k \to y - nx$, and then $x_k + nx \to y$, so H is dense in $\mathbb{R}$. Of course, by a symmetric argument, it is also enough to prove that H is dense in $\mathbb{R}^- = (-\infty, 0]$.

Now we define

$$\begin{aligned} a &= \inf\{x \in H : x > 0\} \\ b &= \sup\{x \in H : x < 0\}. \end{aligned}$$

If $a = 0$ then there exists a sequence $\{x_k\} \subseteq H$ such that $x_k > 0$ for all k and $x_k \to 0$. But then the set $\{n \cdot x_k : n, k \in \mathbb{N}\} \subseteq H$ is dense in $\mathbb{R}^+$, so we are done. Similarly, if $b = 0$ then H is dense in $\mathbb{R}^-$ and we are done as well. Hence we will assume that $b < 0 < a$, and prove that H is cyclic.

Our next goal is to prove that $|a| = |b|$. If not, we may assume without loss of generality that $|a| > |b|$, or, put differently, $a + b > 0$. Choose sequences $\{x_k\}, \{y_k\} \subseteq H$ such that $x_k \to a$ and $y_k \to b$. Then $x_k + y_k \to a + b$. Since

$0 < a + b < a$, for large enough k we have $0 < x_k + y_k < a$, which is a contradiction to the definition of a. Therefore $b = -a$ like we wanted.

Now we prove that $a \in H$. If $a \notin H$, then for every $\varepsilon > 0$ one can find an element $x \in H$ such that $a < x < a + \varepsilon$. In particular, we can choose x such that $a < x < 2a$. Like before, choose a sequence $\{y_k\} \subseteq H$ such that $y_k \to b = -a$. Then $x + y_k \to x - a$, and in particular for large enough k we have $0 < x + y_k < a$. Again, this is a contradiction to the definition of a, so $a \in H$. An identical argument shows that $-a \in H$ as well.

To conclude the proof we finally show that $H = \langle a \rangle = \{na : n \in \mathbb{Z}\}$. The fact that $\langle a \rangle \subseteq H$ is now obvious. For the other direction, every element $x \in \mathbb{R}^+$ can be written as $x = na + y$ for $n \in \mathbb{N}$ and $0 \leq y < a$. If $x \in H$ then

$$y = x - na = x + n(-a) \in H$$

as well, so by the minimality of a we must have $y = 0$. Therefore $x = na$, so $x \in \langle a \rangle$. A similar argument works in the case $x \in \mathbb{R}^-$, and the proof is complete. $\square$

Proof of Lemma 6. One of the implications is simple: If $p_{\ell_1}, p_{\ell_2}, p_{\ell_3}$ are collinear then we can write

$$p_{\ell_3} = \lambda p_{\ell_1} + (1 - \lambda) p_{\ell_2}$$

for some $\lambda \in \mathbb{R}$. This implies that

$$\ell_3 = \lambda \ell_1 + (1 - \lambda) \ell_2,$$

and then $\ell_1(x_0) = \ell_2(x_0) = c$ implies $\ell_3(x_0) = \lambda c + (1 - \lambda)c = c$ as well.

The other implication is almost as easy. Write

$$\ell_i(x) = \langle x, a_i \rangle + c_i$$

for $i = 1, 2, 3$. Our assumption can be reformulated as saying that $\langle x, a_1 - a_2 \rangle = c_2 - c_1$ implies $\langle x, a_1 - a_3 \rangle = c_3 - c_1$. By standard linear algebra, this can only happen if there exists a $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} a_1 - a_3 &= \lambda(a_1 - a_2) \\ c_3 - c_1 &= \lambda(c_2 - c_1). \end{aligned}$$

This is equivalent to

$$\ell_3 = (1 - \lambda)\ell_1 + \lambda \ell_2,$$

which implies collinearity of $p_{\ell_1}, p_{\ell_2}, p_{\ell_3}$ like we wanted. $\square$

3. MEAN WIDTH FOR α-CONCAVE FUNCTIONS

We will now begin our discussion of α-concave functions (see Definition 1). For simplicity, we will restrict ourselves to the case $-\infty < \alpha \leq 0$. For any such α, define

$$C_\alpha(\mathbb{R}^n) = \{f : \mathbb{R}^n \to [0, \infty) : f \text{ is } \alpha\text{-concave, upper semicontinuous and } f \not\equiv 0\}.$$

For example, we have $C_0(\mathbb{R}^n) = \mathrm{LC}(\mathbb{R}^n)$. As stated in section 1, we have $C_{\alpha_1}(\mathbb{R}^n) \subseteq C_{\alpha_2}(\mathbb{R}^n)$ whenever $\alpha_1 \geq \alpha_2$.

Remark 7. In [5], Borell defines not only α-concave functions, but also the notion of a κ-concave measure. A Radon measure μ on $\mathbb{R}^n$ is κ-concave if for any non empty Borel sets A, B and any $0 < \lambda < 1$ we have

$$\mu\left(\lambda A + (1 - \lambda)B\right) \geq \left[\lambda\mu(A)^{\kappa} + (1 - \lambda)\mu(B)^{\kappa}\right]^{\frac{1}{\kappa}}.$$

Borell then proves that α-concave functions and κ-concave measures are closely related: Assume μ is not supported on any hyperplane. Then μ is κ-concave if and only if $\kappa \leq \frac{1}{n}$, μ is absolutely continuous with respect to the Lebesgue measure, and the density $f = \frac{d\mu}{dx}$ is α-concave, for $\alpha = \frac{\kappa}{1-n\kappa}$.

Notice that for such a density $f = \frac{d\mu}{dx}$, we must have $\alpha \geq -\frac{1}{n}$, so some authors only discuss α-concave functions for such values of α. We will need the assumption $\alpha \geq -\frac{1}{n}$ for some theorems, but other results will hold in full generality.

Since we only care about negative values of α, it will often be more convenient, and less confusing, to use the parameter $\beta = -\frac{1}{\alpha}$. For example, we will use the new notation in the following definition:

Definition 8. The convex base of the a function $f \in C_{\alpha}(\mathbb{R}^n)$ is

$$\text{base}_{\alpha}(f) = \frac{1 - f^{\alpha}}{\alpha}.$$

Put differently, $\varphi = \text{base}_{\alpha}(f)$ is the unique convex function such that

$$f = \left(1 + \frac{\varphi}{\beta}\right)^{-\beta}.$$

The above definition is inspired by the work of Bobkov in [4]. While the definition might seem unintuitive at first, it has a couple of appealing features :

- In the limiting case $\alpha \to 0$ ($\beta \to \infty$), we get the relation $\text{base}_0 f = -\log f$. This is the standard and often used bijective, order reversing map between $LC(\mathbb{R}^n)$ and $\text{Cvx}(\mathbb{R}^n)$.
- If $f = \mathbf{1}_K$ is an indicator function, then

$$(\text{base}_{\alpha} f)(x) = \mathbf{1}_K^{\infty}(x) = \begin{cases} 0 & x \in K \\ \infty & \text{otherwise} \end{cases}$$

  is the well known "convex indicator function" of K. In particular, $\text{base}_{\alpha} f$ is independent of α in that case.

Notice, however, that unlike the log-concave case, the map $\text{base}_{\alpha} : C_{\alpha}(\mathbb{R}^n) \to \text{Cvx}(\mathbb{R}^n)$ is not surjective, as we always have $\text{base}_{\alpha} f > -\beta$.

If we are willing to treat $\text{base}_{\alpha} f$ as the proper generalization of $(-\log f)$ to the α-concave case, a few important definitions emerge immediately:

Definition 9.   (1) The α-support function of a function $f \in C_{\alpha}(\mathbb{R}^n)$ is

$$h_f^{(\alpha)} = (\text{base}_{\alpha} f)^* \in \text{Cvx}(\mathbb{R}^n).$$

(2) The α-sum of two functions $f, g \in C_\alpha(\mathbb{R}^n)$ is defined by

$$\text{base}_\alpha(f \star_\alpha g) = (\text{base}_\alpha f) \,\square\, (\text{base}_\alpha g),$$

assuming the right hand side is a indeed a convex base for an α-concave function (see the discussion above Proposition 10). Here $\square : \text{Cvx}(\mathbb{R}^n) \times \text{Cvx}(\mathbb{R}^n) \to \text{Cvx}(\mathbb{R}^n)$ is the standard inf-convolution, defined by

$$(\varphi \square \psi)(x) = \inf_{y+z=x}[\varphi(y) + \psi(z)].$$

(3) If $f \in C_\alpha(\mathbb{R}^n)$ and $\lambda > 0$, the λ-homothety of f defined by

$$[\text{base}_\alpha(\lambda \cdot_\alpha f)](x) = \lambda \cdot (\text{base}_\alpha f)\left(\frac{x}{\lambda}\right).$$

When there is no cause for confusion, we will omit the script and write h_f, $f \star g$, and $\lambda \cdot f$.

The above definitions were constructed to interact well with one another. We have for example

$$h^{(\alpha)}_{(\lambda \cdot_\alpha f) \star_\alpha g} = \lambda h^{(\alpha)}_f + h^{(\alpha)}_g,$$

as well as $f \star_\alpha f = 2 \cdot_\alpha f$ and other similar equalities. However, one should be aware of two important caveats.

The first thing to observe is that the above definitions really depend on α. We know that if $f \in C_{\alpha_1}(\mathbb{R}^n)$, then $f \in C_{\alpha_2}(\mathbb{R}^n)$ for every $\alpha_2 < \alpha_1$. Nonetheless, we usually have $h^{(\alpha_1)}_f \neq h^{(\alpha_2)}_f$, and similarly for additions and homotheties. An important exception to this rule is the case of indicator functions. If $f = \mathbf{1}_K$ and $g = \mathbf{1}_T$, then $h^{(\alpha)}_f = h_K$, $f \star_\alpha g = \mathbf{1}_{K+T}$ and $\lambda \cdot_\alpha f = \mathbf{1}_{\lambda K}$, for all values of α.

The second, technical, caveat is that additions and homotheties are not always defined. If, for example, $\text{base}_\alpha f = \text{base}_\alpha g = -\frac{3\beta}{4}$, then

$$\text{base}_\alpha(f \star_\alpha g) = (\text{base}_\alpha f) \,\square\, (\text{base}_\alpha g) = -\frac{3\beta}{2}.$$

But this is impossible, since for every $h \in C_\alpha(\mathbb{R}^n)$ we have $\text{base}_\alpha h \geq -\beta$. Addition is defined, however, under some mild conditions on f and g (for example it is enough to assume $f \leq 1$). A particularly nice case is the case of convex combinations, where we have the following simple formula:

Proposition 10. *Fix $f, g \in C_\alpha(\mathbb{R}^n)$ and choose $0 < \lambda < 1$. Define*

$$h = [\lambda \cdot f] \star [(1 - \lambda) \cdot g].$$

Then

$$h(x) = \sup_{y+z=x}\left[\lambda f\left(\frac{y}{\lambda}\right)^\alpha + (1-\lambda)g\left(\frac{z}{1-\lambda}\right)^\alpha\right]^{\frac{1}{\alpha}}$$

Proof. This is nothing more than an explicit calculation. Denote $\varphi = \text{base}_\alpha f$ and $\psi = \text{base}_\alpha g$. Then:

$$
\begin{aligned}
h(x)^\alpha &= \left(1 + \frac{\text{base}_\alpha h}{\beta}\right)^{-\beta\alpha} \\
&= 1 + \inf_{y+z=x} \frac{\lambda\varphi\left(\frac{y}{\lambda}\right) + (1-\lambda)\psi\left(\frac{z}{1-\lambda}\right)}{\beta} \\
&= \inf_{y+z=x}\left[\lambda\left(1 + \frac{\varphi\left(\frac{y}{\lambda}\right)}{\beta}\right) + (1-\lambda)\left(1 + \frac{\psi\left(\frac{z}{1-\lambda}\right)}{\beta}\right)\right] \\
&= \inf_{y+z=x}\left[\lambda f\left(\frac{y}{\lambda}\right)^\alpha + (1-\lambda)g\left(\frac{z}{1-\lambda}\right)^\alpha\right],
\end{aligned}
$$

and raising both sides to power $\frac{1}{\alpha}$ we get the result. $\qquad\square$

Proposition 10 is especially useful when combined with a known inequality, discovered independently by Borell ([5]) and Brascamp and Lieb ([6]):

Theorem (Borell-Brascamp-Lieb). *Assume we are given measurable functions $f, g, h : \mathbb{R}^n \to [0,\infty]$ and numbers $0 < \lambda < 1$, $\alpha \geq -\frac{1}{n}$ such that*

$$h\left(\lambda x + (1-\lambda)y\right) \geq \left[\lambda f(x)^\alpha + (1-\lambda)g(y)^\alpha\right]^{\frac{1}{\alpha}}$$

whenever $f(x), g(y) > 0$. Then

$$\int h \geq \left[\lambda\left(\int f\right)^\kappa + (1-\lambda)\left(\int g\right)^\kappa\right]^{\frac{1}{\kappa}},$$

where $\kappa = \frac{\alpha}{1+n\alpha}$.

The importance of the parameter κ was explained in Remark 7. Notice that when $\alpha = \infty$ we get $\kappa = \frac{1}{n}$ and the theorem reduces to the Brunn-Minkowski theorem. When $\alpha = 0$ we get that $\kappa = 0$ as well and the theorem reduces to the a special case known as the Prékopa–Leindler inequality.

From Proposition 10 and the preceding theorem we immediately get:

Corollary 11. *If $f, g \in C_\alpha(\mathbb{R}^n)$ and $\alpha \geq -\frac{1}{n}$, then*

$$\int \left[\lambda \cdot f\right] \star \left[(1-\lambda) \cdot g\right] \geq \left[\lambda\left(\int f\right)^\kappa + (1-\lambda)\left(\int g\right)^\kappa\right]^{\frac{1}{\kappa}}.$$

Our next goal is to define the mean width of an α-concave function. For log-concave functions, the concept of mean width was originally defined by Klartag and Milman in [10]. If $f \in \text{LC}(\mathbb{R}^n)$, the Klartag-Milman definition for the mean width of f is, up to some universal constant,

$$w(f) = \lim_{\varepsilon \to 0^+} \frac{\int G \star [\varepsilon \cdot f] - \int G}{\varepsilon},$$

where $G(x) = e^{-|x|^2/2}$ is the (unnormalized) Gaussian. Since we are dealing with log-concave functions, $\star$ means $\star_0$ in our notation, and similarly for the homothety

operation. In [11], the author presented an equivalent definition, as the average of the support function with respect to the Gaussian measure:

$$w(f) = \int_{\mathbb{R}^n} h_f(x) \cdot G(x) dx.$$

Both definitions can be extended, mutatis mutandis, to general α-concave functions.

Definition 12.   (1) For $-\infty < \alpha \leq 0$ define a function $G_\alpha \in C_\alpha(\mathbb{R}^n)$ by

$$G_\alpha(x) = \left(1 + \frac{|x|^2}{2\beta}\right)^{-\beta},$$

where, as usual $\beta = -\frac{1}{\alpha}$. In other words, we choose G_α to satisfy $\text{base}_\alpha G_\alpha = \frac{|x|^2}{2}$.

(2) For $f \in C_\alpha(\mathbb{R}^n)$ we define its α- mean width as

$$w_\alpha(f) = \lim_{\varepsilon \to 0^+} \frac{\int G_\alpha \star_\alpha [\varepsilon \cdot_\alpha f] - \int G_\alpha}{\varepsilon}$$

The results of [11] can be extended to our case as well. For example we have the following representation formula:

Theorem 13. *For every $f \in C_\alpha(\mathbb{R}^n)$ we get*

$$w_\alpha(f) = \int h_f^{(\alpha)}(x) \cdot \left(1 + \frac{|x|^2}{2\beta}\right)^{-\beta-1} dx$$

Proof. A considerable amount of work is needed in order to write down a completely formal proof, which applies to all cases. All of the details appear in [11] for the log-concave case, but the same strategy works just as well for the general α-concave case. Here we give the essence of the proof, and the suspicious reader may consult [11] for the finer details:

Denote $\varphi = \text{base}_\alpha f$. Then

$$
\begin{aligned}
\text{base}_\alpha\left(G_\alpha \star (\varepsilon \cdot f)\right)(x) &= \inf_y \left[\frac{|x-y|^2}{2} + \varepsilon\varphi\left(\frac{y}{\varepsilon}\right)\right] \\
&= \frac{|x|^2}{2} + \inf_y \left[\frac{|y|^2}{2} - \langle x, y\rangle + \varepsilon\varphi\left(\frac{y}{\varepsilon}\right)\right] \\
&\underset{y=\varepsilon z}{=} \frac{|x|^2}{2} + \inf_z \left[\frac{|\varepsilon z|^2}{2} - \langle x, \varepsilon z\rangle + \varepsilon\varphi(z)\right] \\
&= \frac{|x|^2}{2} + \varepsilon \cdot \inf_z \left[\varepsilon\frac{|z|^2}{2} - \langle x, z\rangle + \varphi(z)\right] \\
&= \frac{|x|^2}{2} - \varepsilon \cdot \sup_z \left[\langle x, z\rangle - \left(\varphi(z) + \varepsilon\frac{|z|^2}{2}\right)\right] \\
&= \frac{|x|^2}{2} - \varepsilon \cdot \varphi_\varepsilon^*(x),
\end{aligned}
$$

where

$$\varphi_\varepsilon(y) = \varphi(y) + \varepsilon \frac{|y|^2}{2}.$$

Define

$$H(x, \varepsilon) = \frac{|x|^2}{2} - \varepsilon \cdot \varphi_\varepsilon^*(x),$$

then by the product rule

$$
\begin{aligned}
\left. \frac{dH}{d\varepsilon} \right|_{\varepsilon=0} &= -\left. \varphi_\varepsilon^*(x) \right|_{\varepsilon=0} - 0 \cdot \left[\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \varphi_\varepsilon^*(x) \right] \\
&= -\varphi^*(x).
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
w_\alpha(f) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int \left(1 + \frac{H(x, \varepsilon)}{\beta} \right)^{-\beta} dx = \int \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \left(1 + \frac{H(x, \varepsilon)}{\beta} \right)^{-\beta} dx \\
&= \int -\beta \left(1 + \frac{H(x, 0)}{\beta} \right)^{-\beta-1} \cdot \frac{1}{\beta} \cdot (-\varphi^*(x)) \, dx \\
&= \int \varphi^*(x) \cdot \left(1 + \frac{|x|^2}{2\beta} \right)^{-\beta-1} dx
\end{aligned}
$$

which is exactly what we wanted. $\qquad\square$

Our next goal is to prove an Urysohn type inequality for $w_\alpha(f)$:

Theorem 14. *If $f \in C_\alpha(\mathbb{R}^n)$ for $\alpha \geq -\frac{1}{n}$ then*

$$w_\alpha(f) \geq \int G_\alpha \cdot \left[\frac{n}{2} + \frac{1}{\kappa} \left(\frac{\int f}{\int G_\alpha} \right)^\kappa - \frac{1}{\kappa} \right],$$

where $\kappa = \frac{\alpha}{1+n\alpha}$

Proof. We can write

$$\int G_\alpha \star [\varepsilon \cdot f] = \int \left[(1 - \varepsilon) \cdot \left(\frac{1}{1 - \varepsilon} \cdot G_\alpha \right) \right] \star [\varepsilon \cdot f],$$

and by Corollary 11 we get

$$\int G_\alpha \star [\varepsilon \cdot f] \geq \left[(1 - \varepsilon) \left(\int \frac{1}{1 - \varepsilon} \cdot G_\alpha \right)^\kappa + \varepsilon \left(\int f \right)^\kappa \right]^{\frac{1}{\kappa}},$$

and the first term in the right hand side can be calculated explicitly:

$$
\begin{aligned}
\int \frac{1}{1 - \varepsilon} \cdot G_\alpha &= \int \left(1 + \frac{|x|^2 (1 - \varepsilon)}{2\beta} \right)^{-\beta} dx \\
&= n\omega_n \int_0^\infty r^{n-1} \left(1 + \frac{r^2(1 - \varepsilon)}{2\beta} \right)^{-\beta} dr \\
&= \left(\frac{2\pi b}{1 - \varepsilon} \right)^{\frac{n}{2}} \frac{\Gamma\left(b - \frac{n}{2} \right)}{\Gamma(b)} = (1 - \varepsilon)^{-\frac{n}{2}} \int G_\alpha.
\end{aligned}
$$

Define

$$A(\varepsilon) \quad = \quad \int G_\alpha \star [\varepsilon \cdot f]$$

$$B(\varepsilon) \quad = \quad \left[(1 - \varepsilon) \left(\int \frac{1}{1 - \varepsilon} \cdot G_\alpha \right)^\kappa + \varepsilon \left(\int f \right)^\kappa \right]^{\frac{1}{\kappa}}$$

$$= \quad \left[(1 - \varepsilon)^{1 - \frac{\kappa n}{2}} \left(\int G_\alpha \right)^\kappa + \varepsilon \left(\int f \right)^\kappa \right]^{\frac{1}{\kappa}}.$$

We know that $A(0) = B(0) = \int G_\alpha$, and $A(\varepsilon) \geq B(\varepsilon)$ for every $\varepsilon \geq 0$. Hence we get

$$w_\alpha(f) = A'(0) \geq B'(0),$$

and by direct computation

$$B'(0) \quad = \quad \frac{1}{\kappa} \left[\left(\int G_\alpha \right)^\kappa \right]^{\frac{1}{\kappa} - 1} \cdot \left[- \left(1 - \frac{\kappa n}{2} \right) \left(\int G_\alpha \right)^\kappa + \left(\int f \right)^\kappa \right]$$

$$= \quad \int G_\alpha \cdot \left[\frac{n}{2} + \frac{1}{\kappa} \left(\frac{\int f}{\int G_\alpha} \right)^\kappa - \frac{1}{\kappa} \right]$$

so we get the result. $\qquad\square$

Notice that in the log-concave case $\kappa \to 0$, and Theorem 14 reduces to the inequality

$$w_0(f) \geq (2\pi)^{\frac{n}{2}} \left[\frac{n}{2} + \log \left(\frac{\int f}{\int G} \right) \right]$$

from [11].

Remark 15. In [7], Colesanti and Fragalà deal with expressions of the form

$$\lim_{\varepsilon \to 0^+} \frac{\int g \star [\varepsilon \cdot f] - \int g}{\varepsilon}$$

where g and f are arbitrary log-concave functions. Among other things, they prove analogs of Theorems 13 and 14, assuming f and g are log-concave functions satisfying several technical assumptions. It is not hard to extend their work to our settings, and obtain results for f and g which are merely α-concave. Since the added difficulties are mostly technical, we will not pursue the matter any further in this paper.

Finally, we will demonstrate how one can obtain Poincaré type inequalities by differentiating Urysohn's inequality. As far as we know this result never appeared in print, even for the log-concave case.

Theorem 16. *Fix $\beta > n$. For any smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ which is bounded from below we have*

$$\int |\nabla \psi(x)|^2 \cdot \left(1 + \frac{|x|^2}{2\beta} \right)^{-\beta - 1} dx \quad \geq \quad \frac{\kappa - 1}{\int G_\alpha} \left[\int \psi(x) \left(1 + \frac{|x|^2}{2\beta} \right)^{-\beta - 1} dx \right]^2$$

$$+ \frac{\beta + 1}{\beta} \cdot \int \psi^2(x) \left(1 + \frac{|x|^2}{2\beta} \right)^{-\beta - 2} dx.$$

(as usual, $\kappa = \frac{\alpha}{1+n\alpha} = \frac{1}{n-\beta}$).

Proof. For $t \geq 0$ define $\varphi_t(x) = \frac{|x|^2}{2} + t \cdot \psi(x)$. Since ψ is bounded from below we know that $\varphi_t > -\beta$ for small enough t. Hence we can define

$$f_t = \left(1 + \frac{\varphi_t}{\beta}\right)^{-\beta},$$

and

$$A(t) = w_\alpha(f_t) = \int \varphi_t^*(x) \cdot \left(1 + \frac{|x|^2}{2\beta}\right)^{-\beta-1} dx$$

$$B(t) = \int G_\alpha \cdot \left[\frac{n}{2} + \frac{1}{\kappa}\left(\frac{\int f_t}{\int G_\alpha}\right)^\kappa - \frac{1}{\kappa}\right].$$

We claim that $A(t) \geq B(t)$ for every (small enough) $t \geq 0$. Indeed, if φ_t happens to be convex, f_t is α-concave and the claim follows from Theorem 14. In the general case, replace φ_t by its convex envelope and notice that $B(t)$ increases, while $A(t)$ stays the same.

By inspecting the proof of Theorem 14 or by direct computation, we see that $A(0) = B(0)$. Let us calculate $A'(0), B'(0)$.

For A, we will use the first variation formula for the Legendre transform

$$\dot{\varphi}_t^*(\nabla\varphi_t(x)) = -\dot{\varphi}_t(x),$$

and by plugging in our φ_t and $t = 0$ we see that

$$\dot{\varphi}_0^*(x) = -\dot{\varphi}_0(x) = -\psi(x),$$

so

$$A'(0) = -\int \psi(x) \left(1 + \frac{|x|^2}{2\beta}\right)^{-\beta-1} dx.$$

For B it is easy to differentiate directly:

$$
\begin{aligned}
B'(t) &= \int G_\alpha \cdot \frac{1}{\kappa} \cdot \kappa \cdot \left(\frac{\int f_t}{\int G_\alpha}\right)^{\kappa-1} \cdot \frac{1}{\int G_\alpha} \cdot \int \dot{f}_t \\
&= \left(\frac{\int f_t}{\int G_\alpha}\right)^{\kappa-1} \cdot \int \left[(-\beta)\left(1 + \frac{\varphi_t}{\beta}\right)^{-\beta-1} \cdot \frac{1}{\beta} \cdot \psi\right] \\
&= -\left(\frac{\int f_t}{\int G_\alpha}\right)^{\kappa-1} \int \psi(x) \left(1 + \frac{\varphi_t(x)}{\beta}\right)^{-\beta-1} dx,
\end{aligned}
$$

and then for $t = 0$ we get

$$B'(0) = -\int \psi(x) \left(1 + \frac{|x|^2}{2\beta}\right)^{-\beta-1} dx.$$

Therefore we have $A'(0) = B'(0)$, as was expected.

It now follows that $A''(0) \geq B''(0)$. In order to calculate $A''(0)$ we will use the second variation formula

$$\ddot{\varphi}_t^*(\nabla\varphi_t(x)) + \ddot{\varphi}_t(x) = \left\langle (\text{Hess}\varphi_t)^{-1} \nabla\dot{\varphi}_t(x), \nabla\dot{\varphi}_t(x) \right\rangle$$

(for a proof of this formula, see for example [8]). Plugging in $t = 0$ we get

$$\ddot{\varphi}_0^* (x) + 0 = \left\langle \mathrm{Id}^{-1} \cdot \nabla \psi(x), \nabla \psi(x) \right\rangle = |\nabla \psi(x)|^2 ,$$

and then

$$A''(0) = \int |\nabla \psi(x)|^2 \cdot \left(1 + \frac{|x|^2}{2\beta} \right)^{-\beta-1} dx.$$

For B, we again have to differentiate directly and get

$$
\begin{aligned}
B''(t) \;=\; & -(\kappa - 1) \left(\frac{\int f_t}{\int G_\alpha} \right)^{\kappa-2} \cdot \frac{1}{\int G_\alpha} \cdot (-1) \cdot \left[\int \psi(x) \left(1 + \frac{\varphi_t(x)}{\beta} \right)^{-\beta-1} dx \right]^2 \\
& - \left(\frac{\int f_t}{\int G_\alpha} \right)^{\kappa-1} \int \psi(x) (-\beta - 1) \left(1 + \frac{\varphi_t(x)}{\beta} \right)^{-\beta-2} \frac{1}{\beta} \cdot \psi(x) dx,
\end{aligned}
$$

or, if we put $t = 0$, we get

$$B''(0) = \frac{\kappa - 1}{\int G_\alpha} \left[\int \psi(x) \left(1 + \frac{|x|^2}{2\beta} \right)^{-\beta-1} dx \right]^2 + \frac{\beta+1}{\beta} \cdot \int \psi^2(x) \left(1 + \frac{|x|^2}{2\beta} \right)^{-\beta-2} dx,$$

and the equality we wanted now follows. $\qquad\square$

Corollary 17. *For any smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ we have*

$$\int |\nabla \psi(x)|^2 \, d\gamma_n(x) \geq \int \psi^2(x) d\gamma_n(x) - \left[\int \psi(x) d\gamma_n(x) \right]^2 ,$$

where $d\gamma_n$ is the standard Gaussian probability measure on $\mathbb{R}^n$.

Proof. This is simply the case $\beta = \infty$ of Theorem 16. By inspecting the proof of Theorem 16 we see that in the case $\beta = \infty$ we do not need ψ to be bounded from below.

When $\beta \to \infty$ we have $\kappa \to 0$, $\frac{\beta+1}{\beta} \to 1$, and

$$\left(1 + \frac{|x|^2}{2\beta} \right)^{-\beta-1} , \left(1 + \frac{|x|^2}{2\beta} \right)^{-\beta-2} \to e^{-\frac{|x|^2}{2}} = G(x).$$

Thus we get

$$\int |\nabla \psi(x)|^2 G(x) dx \geq \int \psi^2(x) G(x) dx - \frac{1}{\int G} \cdot \left[\int \psi(x) G(x) dx \right]^2 ,$$

and if divide both sides by $\int G$ we get exactly what we wanted. $\qquad\square$

We see that Theorem 16 implies the Gaussian Poincaré inequality, with a sharp constant. Hence, the case of general β may be considered as a "generalized Poincaré inequality". At the moment we are not aware of any applications for this generalized form.

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