Characterization of Self-Polar Convex Functions

Liran Rotem

School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel

Abstract

In a work by Artstein-Avidan and Milman the concept of polarity is generalized from the class of convex bodies to the larger class of convex functions. While the only self-polar convex body is the Euclidean ball, it turns out that there are numerous self-polar convex functions. In this work we give a complete characterization of all rotationally invariant self-polar convex functions on $\mathbb{R}^n$.

Keywords: convexity, polarity
2010 MSC: 52A41, 26A51, 46B10

1. Introduction

One of the most important concepts in convex geometry is that of polarity. Denote by $\mathcal{K}_0^n$ the family of all closed, convex sets $K \subseteq \mathbb{R}^n$ such that $0 \in K$. If $K \in \mathcal{K}_0^n$ we define its polar (or dual) body as

$$K^\circ = \{x \in \mathbb{R}^n : \ \langle x, y \rangle \le 1 \text{ for every } y \in K\} \in \mathcal{K}_0^n,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. It is easy to see that polarity is order reversing, which means that if $K_1 \subseteq K_2$ then $K_1^\circ \supseteq K_2^\circ$. Polarity is also an involution - for every $K \in \mathcal{K}_0^n$ we have $(K^\circ)^\circ = K$. It turns out that these two properties characterize polarity uniquely, as the next theorem shows:

Theorem. *Let $n \ge 2$ and $\mathcal{T} : \mathcal{K}_0^n \to \mathcal{K}_0^n$ satisfy:*

(i) $\mathcal{T}\mathcal{T}K = K$ for all K.

(ii) If $K_1 \subseteq K_2$ then $\mathcal{T}K_1 \supseteq \mathcal{T}K_2$.

Then $\mathcal{T}K = B(K^\circ)$ where $B \in GL_n$ is a symmetric linear transformation.

Email address: `liranro1@post.tau.ac.il` (Liran Rotem)

This theorem was proven by Artstein-Avidan and Milman in [3], but similar theorems on different classes of convex bodies were proven earlier by Gruber in [7] and by Böröczky and Schneider in [6].

When dealing with polarity, the Euclidean ball $D_n \subseteq \mathbb{R}^n$ often plays a special role. The fundamental result here is that D_n is the only self-polar convex body: $D_n^\circ = D_n$, and a very simple proof shows that D_n is the only body with this property.

One example of the importance of D_n when dealing with polarity is the famous Blaschke-Santaló inequality. It states that if K is a symmetric convex body (i.e. If $K = -K$), then

$$|K| \cdot |K^\circ| \leq |D_n| \cdot |D_n^\circ| = |D_n|^2 .$$

Here $|\cdot|$ denotes the Lebesgue volume, and equality holds if and only if K is a linear image of D_n. There exists a generalized version of the inequality for non-symmetric bodies, but we will not need it here. The interested reader may consult [8].

In recent years there was a surge of interesting results concerning generalizations of various concepts from the realm of convex bodies to the realm of convex (or, equivalently, log-concave) functions. Our main object of interest will be $\mathrm{Cvx}_0(\mathbb{R}^n)$, the class of all convex, lower semicontinuous functions $\varphi : \mathbb{R}^n \to [0, \infty]$ satisfying $\varphi(0) = 0$. Notice that we have an order reversing embedding of $\mathcal{K}_0^n$ into $\mathrm{Cvx}_0(\mathbb{R}^n)$, sending K to

$$\mathbf{1}_K^\infty(x) = \begin{cases} 0 & x \in K \\ \infty & \text{otherwise.} \end{cases}$$

A natural question is whether one can extend the concept of polarity from $\mathcal{K}_0^n$ to $\mathrm{Cvx}_0(\mathbb{R}^n)$. The answer to this question is “yes”, as the following theorem by Artstein-Avidan and Milman ([4]) shows:

Theorem. *Let $n \geq 2$ and $\mathcal{T} : Cvx_0(\mathbb{R}^n) \to Cvx_0(\mathbb{R}^n)$ satisfy:*

(i) $\mathcal{T}\mathcal{T}\varphi = \varphi$ for all φ.

(ii) If $\varphi_1 \leq \varphi_2$ then $\mathcal{T}\varphi_1 \geq \mathcal{T}\varphi_2$ (here and after, $\varphi_1 \geq \varphi_2$ means $\varphi_1(x) \geq \varphi_2(x)$ for all x).

Then there exists a symmetric linear transformation $B \in GL_n$ and $c > 0$ such that either:

(a) $\mathcal{T}\varphi = \varphi^* \circ B$, where φ^* is the classic Legendre transform of φ, defined by

$$\varphi^*(x) = \sup_{y \in \mathbb{R}^n} \left[\langle x, y \rangle - \varphi(y) \right]$$

or

(b) $\mathcal{T}\varphi = (c \cdot \varphi^\circ) \circ B$, *where* φ° *is the new Polarity transform of* φ, *defined by*

$$\varphi^\circ(x) = \begin{cases} \sup_{\{y \in \mathbb{R}^n :\ \varphi(y) > 0\}} \frac{\langle x,y \rangle - 1}{\varphi(y)} & x \in \left\{\varphi^{-1}(0)\right\}^\circ \\ \infty & x \notin \left\{\varphi^{-1}(0)\right\}^\circ . \end{cases}$$

Even though we have two essentially different order reversing involutions, only the polarity transform extends the classical notion of duality, in the sense that

$$\left(\mathbf{1}_K^\infty\right)^\circ = \mathbf{1}_{K^\circ}^\infty .$$

Therefore it makes sense to think about φ° as the polar function to φ.

Once we have extended the definition of polarity to convex functions, we want to extend our theorems as well. A functional version of the Blaschke-Santaló inequality was proven by Ball in [5]: It follows from his work that if $\varphi \in \mathrm{Cvx}_0\left(\mathbb{R}^n\right)$ is an even function (i.e. $\varphi(x) = \varphi(-x)$), then

$$\int_{\mathbb{R}^n} e^{-\varphi(x)} dx \cdot \int_{\mathbb{R}^n} e^{-\varphi^*(x)} dx \le \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} dx \right)^2 = (2\pi)^n .$$

Again, there is a generalization for the non-even case, proven by Artstein-Avidan, Klartag and Milman in [2]. In the same paper is it also shown that if φ is a maximizer of the Santaló product, then, up to a linear transformation, we must have $\varphi = \frac{|x|^2}{2}$. Since one can easily check that $\left(\frac{|x|^2}{2}\right)^* = \frac{|x|^2}{2}$ and $\frac{|x|^2}{2}$ is the only function with this property, we get once again that the maximizer in the Santaló inequality is the unique self-dual function.

Rather surprisingly, the above mentioned theorem seems to use the "wrong" notion of polarity. It would be interesting have an analogous theorem for φ°, that is to find the maximizer of

$$\int_{\mathbb{R}^n} e^{-\varphi(x)} dx \cdot \int_{\mathbb{R}^n} e^{-\varphi^\circ(x)} dx.$$

Given the classical and the functional Santaló inequalities, it makes sense to conjecture that the maximizer here will be self-polar as well, that is $\varphi = \varphi^\circ$. An independent argument by Artstein-Avidan ([1]) proves that the maximizer must be rotationally invariant, i.e. of the form $\varphi(x) = \rho(|x|)$ for a convex function $\rho : [0, \infty) \to [0, \infty]$. Therefore we are naturally led to the following question: what are all the self-polar, rotationally invariant convex functions?

In order to answer this question we first follow [4] and observe that if $\varphi(x) = \rho\left(|x|\right)$ then $\varphi^\circ(x) = \rho^\circ(|x|)$, where ρ° is the polarity transform of ρ on the ray (see the next section for an exact definition). Therefore it is enough to find all self-polar functions $\varphi : [0, \infty) \to [0, \infty]$. An infinite family of such functions is easy to present: For every $1 \le p \le \infty$ the function

$$\varphi_p(x) = \sqrt{\frac{(p-1)^{p-1}}{p^p}} \cdot x^p$$

is self-polar. Our main result in this paper is that there are, in fact, many other 1-dimensional self-polar functions. Specifically, Theorem 5 provides a complete characterization of self-polar functions on the ray.

Unfortunately, because the set of self-polar functions is so big, finding the Santaló maximizer inside this set seems rather intractable at the moment, and the original question we started with remains open. Nevertheless, we believe the results presented here are of independent interest, and might have applications in several directions.

2. Self-polar functions on the ray

Let $\mathrm{Cvx}_0(\mathbb{R}^+)$ be the class of all convex, lower semicontinuous functions $\varphi : [0,\infty) \to [0,\infty]$ satisfying $\varphi(0) = 0$. For a function $\varphi \in \mathrm{Cvx}_0(\mathbb{R}^+)$, we define its polar φ° as

$$\varphi^\circ(x) = \sup_{y>0} \frac{xy-1}{\varphi(y)}.$$

The division in the definition is formal, in the sense that $\varphi(y)$ may be equal to 0. We remedy the situation by defining "$\frac{+}{0} = \infty$" and "$\frac{-}{0} = 0$". Put differently, we define $\varphi^\circ(x) = \infty$ whenever there exists a $y \in \mathbb{R}^+$ such that $\varphi(y) = 0$ and $xy - 1 > 0$ (or, in other words, whenever $x \notin \left\{\varphi^{-1}(0)\right\}^\circ$).

Just like in the n-dimensional case, polarity on the ray is also an order reversing involution. Our main goal is to characterize all functions $\varphi \in \mathrm{Cvx}_0(\mathbb{R}^+)$ such that $\varphi = \varphi^\circ$.

Definition 1. Denote by F the concave function $F(x) = \sqrt{x^2-1}$ (defined for $x \geq 1$). For $1 \leq q < \infty$, we define

$$T_q = \left\{ \varphi \in \mathrm{Cvx}_0\left(\mathbb{R}^+\right) : \begin{array}{l} \varphi(x) \geq F(x) \text{ for all } x \geq 1 \\ \varphi(q) = F(q) \end{array} \right\}.$$

In other words, T_q is the set of functions which are tangent to F at q. For $q = \infty$ we define

$$T_\infty = \left\{ \varphi \in \mathrm{Cvx}_0\left(\mathbb{R}^+\right) : \begin{array}{l} \varphi(x) \geq F(x) \text{ for all } x \geq 1 \\ \lim_{x\to\infty}\left(\varphi(x) - F(x)\right) = 0 \end{array} \right\}.$$

The classes T_1 and T_∞ will be exceptional and somewhat trivial. In fact, let us define the following:

Definition 2. For $\beta \in \mathbb{R}$ we define a function $\ell_\beta \in \mathrm{Cvx}_0(\mathbb{R}^+)$ by $\ell_\beta(x) = \beta x$. In other words, ℓ_β is the line through the origin with slope β.

Using this definition, it is not hard to see that $T_1 = \left\{ \mathbf{1}_{[0,1]}^{\infty} \right\}$, and $T_\infty = \{\ell_1\}$. For $1 < q < \infty$, the class T_q is infinite.

Our first proposition will explain the importance of these classes when characterizing self-polar functions:

Proposition 3.

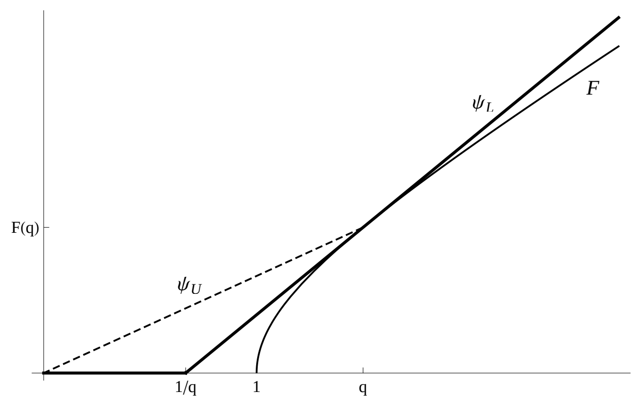
(i) If $\varphi \in T_q$ for some $1 \le q \le \infty$, then $\varphi^\circ \in T_q$.

(ii) If $\varphi = \varphi^\circ$, then $\varphi \in T_q$ for some $1 \le q \le \infty$.

Proof. (i) If $q = 1, \infty$ this is trivial by the above comment. For $1 < q < \infty$ define two convex functions ψ_L and ψ_U as

$$\begin{aligned} \psi_L &= \mathbf{1}_{\left[0,\frac{1}{q}\right]}^{\infty} \wedge \ell_{F'(q)} \\ \psi_U &= \mathbf{1}_{[0,q]}^{\infty} \vee \ell_{F(q)/q}. \end{aligned}$$

Here and after, $\vee$ and $\wedge$ will denote supremum and infimum in the lattice $\mathrm{Cvx}_0(\mathbb{R}^+)$. In other words, $\varphi_1 \vee \varphi_2 = \max(\varphi_1, \varphi_2)$, and $\varphi_1 \wedge \varphi_2$ is the biggest function in $\mathrm{Cvx}_0(\mathbb{R}^+)$ which is smaller than $\min(\varphi_1, \varphi_2)$. To illustrate these definitions we plot the graphs of ψ_L and ψ_U:



It is clear that $\varphi \in T_q$ if and only if $\psi_L \le \varphi \le \psi_U$. Since polarity is order reversing, we get that if $\varphi \in T_q$ then $\psi_U^\circ \le \varphi^\circ \le \psi_L^\circ$. But

$$\psi_L^\circ = \left(\mathbf{1}_{\left[0,\frac{1}{q}\right]}^{\infty} \right)^\circ \vee \left(\ell_{F'(q)} \right)^\circ = \mathbf{1}_{[0,q]}^{\infty} \vee \ell_{1/F'(q)} = \psi_U,$$

and thus $\psi_U^\circ = \psi_L$, so $\varphi^\circ \in T_q$ as well.

(ii) First notice that if $\varphi = \varphi^\circ$ then for every $x \ge 0$ we have

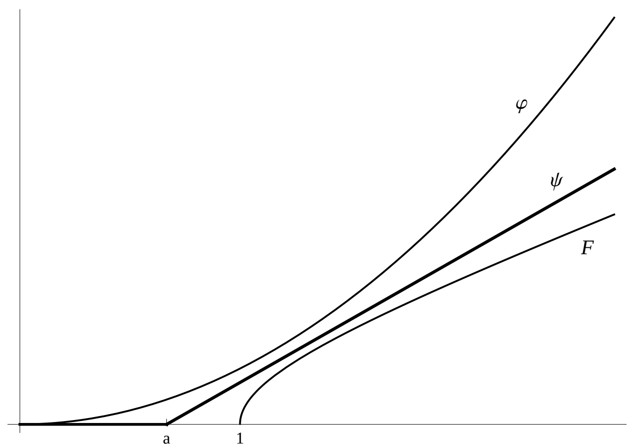
$$\varphi(x) = \varphi^\circ(x) = \sup_{y>0} \frac{xy - 1}{\varphi(y)} \ge \frac{x^2 - 1}{\varphi(x)},$$

and if $x \geq 1$ this implies $\varphi(x) \geq F(x)$.

Define convex sets in $\left(\mathbb{R}^+\right)^2$ by

$$\begin{aligned} C_1 &= epi(\varphi) = \left\{(x,y) \in \left(\mathbb{R}^+\right)^2 : \ y \geq \varphi(x)\right\} \\ C_2 &= hyp(F) = \left\{(x,y) \in \left(\mathbb{R}^+\right)^2 : \ y \leq F(x)\right\}. \end{aligned}$$

If $d\left(C_1, C_2\right) = 0$ then $\varphi \in T_q$ for some q (which can be ∞), so we will assume by contradiction that $d(C_1, C_2) > 0$. This means that there is a line ℓ strictly separating C_1 and C_2 (see, e.g. Theorem 11.4 in [9]). Denote by β the slope of ℓ and by a the intersection of ℓ and the $x-$axis. Define $\psi = \mathbf{1}_{[0,a]}^{\infty} \wedge \ell_\beta$:



On the one hand, we know that

$$\psi \leq \varphi = \varphi^\circ \leq \psi^\circ = \mathbf{1}_{[0,a^{-1}]}^{\infty} \vee \ell_{\beta^{-1}},$$

so in particular $\psi\left(\frac{1}{a}\right) \leq \psi^\circ\left(\frac{1}{a}\right)$. More explicitly, this means

$$\beta\left(\frac{1}{a} - a\right) \leq \frac{1}{\beta} \cdot \frac{1}{a},$$

or $\beta^2\left(1 - a^2\right) \leq 1$.

On the other hand, it is easy to compute that the tangent to F passing through $(a, 0)$ has slope $\frac{1}{\sqrt{1-a^2}}$. Since $\psi > F$ we must have $\beta > \frac{1}{\sqrt{1-a^2}}$, or $\beta^2(1-a^2) > 1$. This is a contradiction, so no such ψ can exist and $\varphi \in T_q$ for some $1 \leq q \leq \infty$. $\square$

Our next goal is to explain how to construct fixed points inside T_q for $1 < q < \infty$. To do so we will need the following proposition, which shows that in order to calculate $\varphi^\circ(x)$ we don't need to know $\varphi(y)$ for all possible values of y:

Proposition 4. *Assume $\varphi_1, \varphi_2 \in T_q$ for some $1 < q < \infty$.*

(i) If $\varphi_1(x) = \varphi_2(x)$ for all $x \le q$, then $\varphi_1^\circ(x) = \varphi_2^\circ(x)$ for all $x \ge q$.

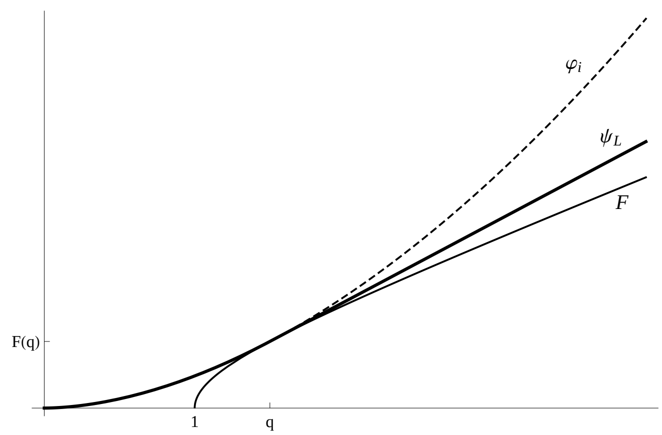
(ii) If $\varphi_1(x) = \varphi_2(x)$ for all $x \ge q$, then $\varphi_1^\circ(x) = \varphi_2^\circ(x)$ for all $x \le q$.

Proof. We will prove (i), and the proof of (ii) is analogous.

Define convex functions ψ_L and ψ_U by

$$\begin{aligned} \psi_U &= \varphi \vee \mathbf{1}_{[0,q]}^{\infty} \\ \psi_L &= \left(\varphi \vee \mathbf{1}_{[0,q]}^{\infty}\right) \wedge \ell_{F'(q)}, \end{aligned}$$

where φ is either φ_1 or φ_2 – the definitions remain the same regardless of this choice. We claim that $\psi_L \le \varphi_i \le \psi_U$ for $i = 1, 2$. The right inequality is obvious. For the left inequality, notice that φ_i and ψ_L coincide for $x \le q$. For $x \ge q$ the function ψ_L grows linearly, and in fact is exactly the tangent line to F at the point q. Since $\varphi_i \in T_q$ we get that ψ_L is also a tangent for φ_i, and because φ_i is convex we must have $\varphi_i \ge \psi_L$:



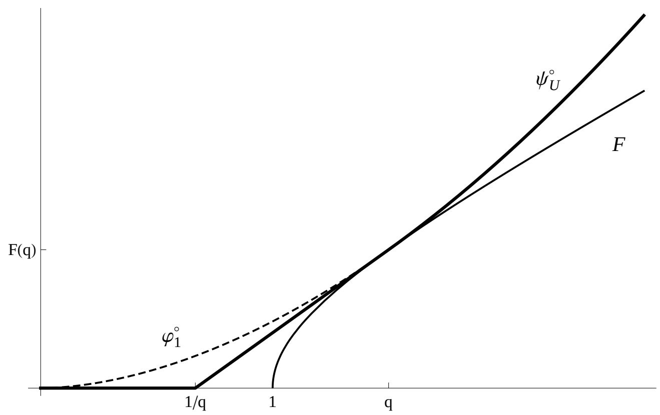
Since polarity is order reversing we get that $\psi_U^\circ \le \varphi_i^\circ \le \psi_L^\circ$, so all we need to show is that $\psi_U^\circ(x) = \psi_L^\circ(x)$ for every $x \ge q$. A direct computation yields

$$\begin{aligned} \psi_U^\circ &= \varphi_1^\circ \wedge \mathbf{1}_{[0,q^{-1}]}^{\infty} \\ \psi_L^\circ &= \left(\varphi_1^\circ \wedge \mathbf{1}_{[0,q^{-1}]}^{\infty}\right) \vee \ell_{F'(q)^{-1}}, \end{aligned}$$

so we need to show that if $x \ge q$ then

$$\frac{x}{F'(q)} \le \left(\varphi_1^\circ \wedge \mathbf{1}_{[0,q^{-1}]}^{\infty}\right)(x).$$

By Proposition 3 we know that $\varphi_1^\circ \in T_q$. In particular, the tangent line to F at q is also a tangent line for φ_1°. But an easy computation shows that this line passes through $\left(q^{-1}, 0\right)$, so we know how ψ_U° looks:



In particular, wee see that if $x \geq q$ then

$$\left(\varphi_1^\circ \wedge \mathbf{1}^\infty_{[0,q^{-1}]}\right)(x) = \varphi_1^\circ(x).$$

Finally, since φ_1° is convex, we know that for every $x \geq q$

$$\frac{\varphi_1^\circ(x)}{x} \geq \frac{\varphi_1^\circ(q)}{q} = \frac{F(q)}{q} = \frac{1}{F'(q)},$$

which implies

$$\frac{x}{F'(q)} \leq \varphi_1^\circ(x)$$

like we wanted. □

Using the last two propositions it is easy to give a complete characterization of 1-dimensional self-polar convex functions:

Theorem 5. *For every $\varphi \in T_q$ define*

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x) & x \leq q \\ \varphi^\circ(x) & x \geq q. \end{cases}$$

Then $\widetilde{\varphi}$ is self-polar, and any self-polar function is of the form $\widetilde{\varphi}$ for some φ.

Proof. Since both φ and φ° are tangent to F at q, the function $\widetilde{\varphi}$ is indeed convex. Using Proposition 4 twice we see that $\widetilde{\varphi}$ is self-polar (compare once with φ, and once with φ°). In the other direction, if φ is self-polar then by Proposition 3 we know that $\varphi \in T_q$ for some q, and then $\varphi = \widetilde{\varphi}$. □

Finally, we would like to state that, rather surprisingly, there are self-polar functions in $\text{Cvx}_0(\mathbb{R}^n)$ which are not rotationally invariant. As one example, it is easy to compute directly that the function $\varphi \in \text{Cvx}_0\left(\mathbb{R}^2\right)$ defined by

$$\varphi(x,y) = \begin{cases} |y| & \text{if } |x| \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

is self-polar. This means that our classification of self-polar functions in $\mathrm{Cvx}_0(\mathbb{R}^+)$ does not give a complete classification of all self-polar functions in $\mathrm{Cvx}_0(\mathbb{R}^n)$.

[1] S. Artstein-Avidan. Steiner symmetrizations and polarity for functions, 2011. preprint.

[2] S. Artstein-Avidan, B. Klartag, and V. Milman. The santaló point of a function, and a functional form of the santaló inequality. *Mathematika*, 51(1-2):33–48, 2004.

[3] S. Artstein-Avidan and V. Milman. The concept of duality for measure projections of convex bodies. *J. Funct. Anal.*, 254(10):2648–2666, 2008.

[4] S. Artstein-Avidan and V. Milman. Hidden structures in the class of convex functions and a new duality transform. *J. Eur. Math. Soc.*, 13(4):975–1004, 2011.

[5] K. Ball. *Isometric problems in ℓ_p and sections of convex sets.* PhD thesis, Cambridge University, 1986.

[6] K. J. Böröczky and R. Schneider. A characterization of the duality mapping for convex bodies. *Geom. Funct. Anal.*, 18(3):657–667, 2008.

[7] P. M. Gruber. The endomorphisms of the lattice of norms in finite dimensions. *Abh. Math. Semin. Univ. Hambg.*, 62(1):179–189, 1992.

[8] M. Meyer and A. Pajor. On the Blaschke-Santaló inequality. *Arch. Math. (Basel)*, 55(1):82–93, 1990.

[9] R. T. Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.