Mixed integrals and related inequalities

Liran Rotem
(joint work with Vitali Milman)

Our point of departure will be Minkowski's theorem on mixed volumes:

Theorem 1 (Minkowski)**.** *Fix bodies* $K_1, K_2, \ldots, K_m \in \mathcal{K}_c^n$*. Then the function* $F : (\mathbb{R}^+)^m \to [0,\infty)$*, defined by*

$$F(\lambda_1, \lambda_2, \ldots, \lambda_m) = \mathrm{Vol}\left(\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m\right),$$

is a homogeneous polynomial of degree n*, with non-negative coefficients.*

Here $\mathcal{K}_c^n$ is the family of compact and convex bodies in $\mathbb{R}^n$, and the addition operation $+$ is Minkowski addition,

$$A + B = \{a + b : \ a \in A, \ b \in B\}.$$

By standard linear algebra, Minkowski's theorem is equivalent to the existence of a map $V : (\mathcal{K}_c^n)^n \to [0,\infty)$ which is multilinear, symmetric and which satisfies $V(K, K, \ldots, K) = \mathrm{Vol}(K)$. This map is unique, and the number $V(K_1, K_2, \ldots, K_n)$ is known as the mixed volume of $K_1, \ldots, K_n$.

Our goal is to extend Minkowski's theorem to a functional setting. That is, we want to take n functions $f_1, f_2, \ldots, f_n : \mathbb{R}^n \to [0,\infty)$ and define their "mixed volume" $V(f_1, f_2, \ldots, f_n)$. In order to do so we need to choose an appropriate family of functions, a "volume" functional on this family, and an addition operation.

For the family of functions, we choose the class of *quasi-concave* functions. A function $f : \mathbb{R}^n \to [0,\infty)$ is called quasi-concave if for every $x, y \in \mathbb{R}^n$ and every $0 < \lambda < 1$ we have

$$f\left(\lambda x + (1-\lambda)y\right) \geq \min\left\{f(x), f(y)\right\}.$$

While not always necessary, it is very convenient to assume further that f is upper semicontinuous, that $\max f = f(0) = 1$ and that $f(x) \to 0$ as $|x| \to 0$. Denote this set of functions by $\mathrm{QC}\left(\mathbb{R}^n\right)$.

As a volume, we choose the Lebesgue integral, i.e.

$$\mathrm{Vol}(f) = \int_{\mathbb{R}^n} f(x)dx.$$

Finally, for addition, we define a new addition on quasi-concave functions by

$$\left(f \oplus g\right)(x) = \sup_{y \in \mathbb{R}^n} \min\left\{f(y), g(x-y)\right\}.$$

We further define the product $\lambda \odot f$ for $f \in \mathrm{QC}\left(\mathbb{R}^n\right)$ and $\lambda > 0$ by $\left(\lambda \odot f\right)(x) = f\left(\frac{x}{\lambda}\right)$. We briefly comment that these operations emerge as a limit of the natural addition operations on α-concave functions. An explanation of this statement appears in [2] and [4].

Under the above definition, we have to following theorem:

Theorem 2. *Fix $f_1, f_2, \dots, f_m \in \mathrm{QC}(\mathbb{R}^n)$. Then the function $F : (\mathbb{R}^+)^m \to [0,\infty]$, defined by*

$$F(\lambda_1, \lambda_2, \dots, \lambda_m) = \int [(\lambda_1 \odot f_1) \oplus (\lambda_2 \odot f_2) \oplus \cdots \oplus (\lambda_m \odot f_m)]$$

is a homogeneous polynomial of degree n, with non-negative coefficients.

The proof of this result appears in [3]. As usual, this is equivalent to the existence of a multilinear, symmetric map $V : \mathrm{QC}(\mathbb{R}^n)^n \to [0,\infty]$ which satisfies $V(f, f, \dots, f) = \int f$. The number $V(f_1, f_2, \dots, f_m)$ will be called the *mixed integral* of $f_1, f_2, \dots, f_m$. The following theorem summarizes some of the basic properties of mixed integrals:

Theorem 3. (1) *$V(K_1, K_2, \dots, K_n) = V(\mathbf{1}_{K_1}, \mathbf{1}_{K_2}, \dots, \mathbf{1}_{K_n})$.*
(2) *If $f_i \geq g_i$ for all i, then $V(f_1, f_2, \dots, f_n) \geq V(g_1, g_2, \dots, g_n)$.*
(3) *V is rotation and translation invariant.*
(4) *Fix $g_{m+1}, \dots, g_n \in \mathrm{QC}(\mathbb{R}^n)$, and define*

$$\Phi(f) = V(f[m], g_{m+1}, \dots, g_n).$$

Φ satisfies a valuation type property: if $f_1, f_2 \in \mathrm{QC}(\mathbb{R}^n)$ and $f_1 \vee f_2 = \max(f_1, f_2) \in \mathrm{QC}(\mathbb{R}^n)$ as well, then

$$\Phi(f_1 \vee f_2) + \Phi(f_1 \wedge f_2) = \Phi(f_1) + \Phi(f_2).$$

Once we have a generalization of the notion of mixed volumes, it makes sense to try and generalize the important inequalities as well. For example, for $f \in \mathrm{QC}(\mathbb{R}^n)$ define its k-th quermassintegral to be

$$W_k(f) = V(\underbrace{f, f, \dots, f}_{n-k \text{ times}}, \underbrace{\mathbf{1}_D, \mathbf{1}_D, \dots, \mathbf{1}_D}_{k \text{ times}}),$$

where D is the unit Euclidean ball. This notion of functional quermassintegrals was discovered independently by Bobkov, Colesanti and Fragalà ([1]). In particular, we have the notion of surface area, defined by $S(f) = nW_1(f)$.

We now want to prove a functional isoperimetric inequality. Unfortunately, it turns out that for general quasi-concave functions it is impossible to give a lower bound for $S(f)$ in terms of $\int f$. Surprisingly, however, it is possible to state a functional extension of the isoperimetric inequality:

Theorem 4. *For every $f \in \mathrm{QC}(\mathbb{R}^n)$ we have $S(f) \geq S(f^*)$, where f^* is the symmetric decreasing rearrangement of f.*

Plugging in $f = \mathbf{1}_K$, we see that this theorem really generalizes the isoperimetric inequality.

Using a slightly more complicated notion of a "generalized rearrangement", it is possible to prove functional versions of most of the classic inequalities: Brunn-Minkowski (and its extension to mixed volumes), Alexandrov-Fenchel, and others. As a special case, we have the following extension of Theorem 4:

Theorem 5. *For every $f_1, f_2, \dots, f_n \in \mathrm{QC}(\mathbb{R}^n)$ we have $V(f_1, f_2, \dots, f_n) \geq V(f_1^*, f_2^*, \dots, f_n^*)$.*

For indicator functions, this reduces to the known statement that for every convex bodies $K_1, K_2, \ldots, K_n$ in $\mathbb{R}^n$ we have

$$V(K_1, K_2, \ldots, K_n) \geq \left(\prod_{i=1}^{n} \mathrm{Vol}(K_i) \right)^{\frac{1}{n}}.$$

Finally, if one is willing to restrict the class of functions, it is possible to prove certain inequalities in a more familiar form. For example, in the class of geometric log-concave functions we have the following Alexandrov type inequalities:

Theorem 6. *Define* $g(x) = e^{-|x|}$. *For every geometric log-concave function* f *and every integers* $0 \leq k < m < n$ *we have*

$$\left(\frac{W_k(f)}{W_k(g)} \right)^{\frac{1}{n-k}} \leq \left(\frac{W_m(f)}{W_m(g)} \right)^{\frac{1}{n-m}},$$

with equality if and only if $f(x) = e^{-c|x|}$ *for some* $c > 0$.

References

[1] S. Bobkov, A. Colesanti, I. Fragalà, *Quermassintegrals of quasi-concave functions and generalized Prékopa-Leindler inequalities*, preprint.

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[3] V. Milman, L. Rotem, *Mixed integrals and related inequalities*, Journal of Functional Analysis **264(2)**, 570-604.

[4] L. Rotem, *Support functions and mean width for α-concave functions*, Advances in Mathematics, to appear.