MIXED INTEGRALS AND RELATED INEQUALITIES

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Abstract. In this paper we define an addition operation on the class of quasi-concave functions. While the new operation is similar to the well-known sup-convolution, it has the property that it polarizes the Lebesgue integral. This allows us to define mixed integrals, which are the functional analogs of the classic mixed volumes.

We extend various classic inequalities, such as the Brunn-Minkowski and the Alexandrov-Fenchel inequality, to the functional setting. For general quasi-concave functions, this is done by restating those results in the language of rearrangement inequalities. Restricting ourselves to log-concave functions, we prove generalizations of the Alexandrov inequalities in a more familiar form.

1. Introduction

One of the fundamental theorems in classic convexity is Minkowski's theorem on mixed volumes. In order to state the theorem, we will need some basic definitions. Denote by $\mathcal{K}^n$ the class of all closed, convex sets in $\mathbb{R}^n$. On $\mathcal{K}^n$ we have the operation of Minkowski addition, defined by

$$K_1 + K_2 = \left\{x_1 + x_2 : \ x_1 \in K_1, \ x_2 \in K_2\right\}.$$

Similarly, if $K \in \mathcal{K}^n$ and $\lambda \geq 0$, we can define the homothet $\lambda \cdot K$ as

$$\lambda \cdot K = \left\{\lambda x : \ x \in K\right\}.$$

Finally, for $K \in \mathcal{K}^n$ define $\mathrm{Vol}(K) \in [0, \infty]$ to be the standard Lebesgue volume of K. Now we can state Minkowski's theorem (see, e.g. [10] for a proof):

Theorem (Minkowski)**.** *Fix $K_1, K_2, \ldots, K_m \in \mathcal{K}^n$. Then the function $F : (\mathbb{R}^+)^m \to [0, \infty]$, defined by*

$$F(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) = Vol(\varepsilon_1 K_1 + \varepsilon_2 K_2 + \cdots + \varepsilon_m K_m),$$

is a homogenous polynomial of degree n, with positive coefficients.

The main goal of this paper is to extend Minkowski's theorem and related inequalities, from the class of convex bodies to the larger classes of log-concave and quasi-concave functions. We will soon give the relevant definitions and the exact statements, but first let us make a few comments about Minkowski's theorem.

First, notice that we did not make the usual assumption that the sets K_i are compact. This is not a problem, as long as we allow our polynomial to attain the

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value $+\infty$ and adopt the convention that $0 \cdot \infty = 0$. Second, by standard linear algebra, Minkowski's theorem is equivalent to the existence of a polarization for the volume form. More explicitly, there exists a function

$$V : (\mathcal{K}^n)^n \to [0, \infty]$$

which is multilinear, symmetric, and satisfies $V(K, K, \ldots, K) = \mathrm{Vol}(K)$. The number $V(K_1, K_2, \ldots, K_n)$ is called the mixed volume of the $K_1, K_2, \ldots, K_n$, and is nothing more than the relevant coefficient of the Minkowski polynomial:

$$\mathrm{Vol}\left(\varepsilon_1 K_1 + \varepsilon_2 K_2 + \cdots + \varepsilon_m K_m\right) = \sum_{i_1, i_2, \ldots, i_n = 1}^{m} \varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_n} \cdot V(K_{i_1}, K_{i_2}, \ldots, K_{i_n}).$$

We would also like to note that more than anything, Minkowski's theorem is a property of the Minkowski addition. To put this comment in perspective, notice that there are several interesting ways to define the sum of convex bodies, and the Minkowski addition is just one of the possibilities. For example, remember that the support function of a convex body $K \in \mathcal{K}^n$ is a 1-homogenous convex function $h_K : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$h_K(y) = \sup_{x \in K} \langle x, y \rangle .$$

Support functions are connected to the Minkowski sum via the relation

$$h_{K+T}(x) = h_K(x) + h_T(x).$$

Similarly, if K and T are convex sets containing the origin, then for every $1 \le p \le \infty$ we can define their L_p-sum $K +_p T$ using the relation

$$h_{K +_p T}(x) = \left(h_K(x)^p + h_T(x)^p\right)^{\frac{1}{p}} .$$

As a second example, remember that if K is a convex body containing the origin, then its polar body is another convex body defined by

$$K^\circ = \{y \in \mathbb{R}^n : \ h_K(y) \le 1\} .$$

For two such bodies K and T, we can define a summation operation by $K \oplus T = (K^\circ + T^\circ)^\circ$.

In both of the above examples we only defined the addition, and not the homothety operation. However, it is a general fact that sufficiently "nice" addition operations on $\mathcal{K}^n$ induce a natural homothety operation. Specifically, for $m \in \mathbb{N}$ one can always define

$$m \cdot K = \underbrace{K + K + \cdots + K}_{m \text{ times}} .$$

It is often the case that for every $K \in \mathcal{K}^n$ and $m \in \mathbb{N}$ there exists a unique body $T \in \mathcal{K}^n$ such that $m \cdot T = K$, and then it is natural to define $\frac{\ell}{m} \cdot K = \ell \cdot T$. Finally, one extends the definition to a general $\lambda > 0$ using some sort of continuity. Because of this construction we will suppress the role of the homotheties in informal discussions, and will talk only about the addition operation. In other words, we adopt the convention that homotheties are always the induced from, and compatible with, the addition operation. It is easy to see that for L_p-sums the induced homothety operation is $\lambda \cdot_p K = \lambda^{\frac{1}{p}} \cdot K$, and for the polar sum the induced homothety is $\lambda \odot K = \lambda^{-1} \cdot K$.

Our examples of addition share some appealing properties of the Minkowski addition. For example, they are all commutative, associative, and with $\{0\}$ serving as an identity element. However, using some simple examples, one may check that volume is no longer a polynomial, if one replaces Minkowski addition by L_p-addition (for $p > 1$) or polar addition, and the same would be true for "most" possible definitions of addition. In fact, in [6] the authors consider the problem of characterizing the Minkowski addition. Roughly speaking, they show that the Minkowski addition is the only operation on convex bodies satisfying a short list of properties, one of which is polynomiality of volume. However, the convention in [6] is that homotheties are always the classic Minkowski homotheties, and not the ones which are induced from the addition operation (an interesting related question is whether there are any addition operations on convex bodies, other than Minkowski addition, such that the induced homothety operation is the classic one).

In recent years, it became apparent that embedding the class $\mathcal{K}^n$ of convex sets into some class of functions $f : \mathbb{R}^n \to [-\infty, \infty]$ can lead to important implications (see the survey [8]). One natural choice is to embed $\mathcal{K}^n$ into the class of convex functions, by mapping each $K \in \mathcal{K}^n$ to its convex indicator function, defined as

$$\mathbf{1}_K^\infty(x) = \begin{cases} 0 & x \in K \\ \infty & \text{otherwise.} \end{cases}$$

This embedding, however, has the technical disadvantage that convex functions are almost never integrable. To remedy the situation, we usually deal with log-concave functions, which are functions $f : \mathbb{R}^n \to [0, \infty)$ of the form $f = e^{-\varphi}$, where φ is a convex function. In particular, every convex set K is mapped to its standard indicator function,

$$\mathbf{1}_K(x) = \begin{cases} 1 & x \in K \\ 0 & \text{otherwise.} \end{cases}$$

To be a bit more formal, we define

$$\mathrm{Cvx}\left(\mathbb{R}^n\right) = \left\{\varphi : \mathbb{R}^n \to (-\infty, \infty] : \ \varphi \text{ is convex and lower semicontinuous } \right\},$$

and then

$$\mathrm{LC}\left(\mathbb{R}^n\right) = \left\{e^{-\varphi} : \ \varphi \in \mathrm{Cvx}\left(\mathbb{R}^n\right)\right\}$$

is the class of log-concave functions. The semi-continuity assumption is just the analog of the assumption that our convex sets are closed.

We would like to find a Minkowski-type theorem for log-concave functions. In order to achieve this goal, we first need to give meaning to the concept of "volume", and the concept of "addition". For volume, we need some functional $I : \mathrm{LC}\left(\mathbb{R}^n\right) \to [0, \infty)$ such that $I(\mathbf{1}_K) = \mathrm{Vol}(K)$ for all convex bodies K. The obvious candidate is the Lebesgue integral,

$$I(f) = \int_{\mathbb{R}^n} f(x)dx.$$

For addition, matters are more complicated. Defining addition on $\mathrm{LC}\left(\mathbb{R}^n\right)$ is, of course, equivalent to defining addition on $\mathrm{Cvx}\left(\mathbb{R}^n\right)$ – for any operation $\oplus$ on $\mathrm{Cvx}\left(\mathbb{R}^n\right)$ we can define an operation on $\mathrm{LC}\left(\mathbb{R}^n\right)$ by $e^{-\varphi} \oplus e^{-\psi} = e^{-(\varphi \oplus \psi)}$. The first attempt at a definition is probably the pointwise addition,

$$\left(\varphi + \psi\right)(x) = \varphi(x) + \psi(x),$$

which transforms to pointwise multiplication for log-concave functions. This definition, however, has many problems, not the least of which is that it does not extend Minkowski addition:

$$\mathbf{1}_K(x)\cdot\mathbf{1}_T(x)=\mathbf{1}_{K\cap T}(x)\neq\mathbf{1}_{K+T}(x).$$

A better definition, and the one that is usually used in applications, is that of inf-convolution:

$$\left(\varphi\Box\psi\right)(x)=\inf_{y\in\mathbb{R}^n}\left[\varphi(y)+\psi(x-y)\right].$$

The corresponding operation for log-concave functions is the so called sup-convolution or Asplund sum, defined by

$$\left(f\star g\right)(x)=\sup_{y\in\mathbb{R}^n}f(y)g(x-y).$$

The sup-convolution generalized the Minkowski addition, in the sense that $\mathbf{1}_K\star\mathbf{1}_T=\mathbf{1}_{K+T}$. However, there is no Minkowski type theorem for this operation. This is easy to see, as the Lebesgue integral is not even homogenous with respect to the sup-convolution: For a general $f\in\mathrm{LC}\left(\mathbb{R}^n\right)$ we do not have

$$\int\left(f\star f\right)=2^n\int f,$$

as one verifies with simple examples.

We will now define another operation on convex functions (and, by extension, on log-concave functions as well):

Definition 1. The sum of convex functions $\varphi,\psi\in\mathrm{Cvx}\left(\mathbb{R}^n\right)$ is

$$\left(\varphi\oplus\psi\right)(x)=\inf_{y\in\mathbb{R}^n}\max\left\{\varphi(y),\psi(x-y)\right\}.$$

Additionally, if $\lambda>0$ we define the product $\lambda\odot\varphi$ as

$$\left(\lambda\odot\varphi\right)(x)=\varphi\left(\frac{x}{\lambda}\right).$$

On the level of log-concave functions, the operation $\oplus$ is defined by

$$\left(f\oplus g\right)(x)=\sup_{y\in\mathbb{R}^n}\min\left\{f(y),g(x-y)\right\}.$$

For Definition 1 to make sense, we need to know that $\varphi\oplus\psi$ is convex whenever φ and ψ are. We will prove this, together with other properties of $\oplus$, in section 2. For now, let us highlight the main features of this operation.

First, one easily checks that $\odot$ really is the homothety operation induced from $\oplus$. In particular, we have $2\odot\varphi=\varphi\oplus\varphi$ for every $\varphi\in\mathrm{Cvx}\left(\mathbb{R}^n\right)$. Second, the operation $\oplus$ extends the Minkowski addition on convex bodies, in the sense that

$$\mathbf{1}_K\oplus\mathbf{1}_T=\mathbf{1}_{K+T}.$$

Third, the operation $\oplus$ is not so different from the more classic inf-convolution $\Box$. In fact, it follows from Proposition 10 that if φ and ψ are positive, convex functions, then

$$\frac{1}{2}\left(\varphi\Box\psi\right)(x)\leq\left(\varphi\oplus\psi\right)(x)\leq\left(\varphi\Box\psi\right)(x),$$

i.e. both operations agree up to a factor of 2. Furthermore, if φ is a positive, convex function and $K \in \mathcal{K}^n$ is any convex set, then

$$(\varphi \oplus \mathbf{1}_K^\infty)(x) = (\varphi \Box \mathbf{1}_K^\infty)(x) = \inf_{y \in K} \varphi(x-y),$$

so in this case both operations are exactly the same.

However, our new addition has one critical advantage over the better known inf-convolution: The volume functional $I(f)$ polarizes with respect to $\oplus$. In other words, we have the following Minkowski-type theorem:

Theorem. *Fix $f_1, f_2, \ldots, f_m \in \mathrm{LC}(\mathbb{R}^n)$. Then the function $F : (\mathbb{R}^+)^m \to [0,\infty]$, defined by*

$$F(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) = \int [(\varepsilon_1 \odot f_1) \oplus (\varepsilon_2 \odot f_2) \oplus \cdots \oplus (\varepsilon_m \odot f_m)]$$

is a homogenous polynomial of degree n, with non-negative coefficients.

In complete analogy with the case of convex bodies, this theorem is equivalent to the existence of a function

$$V : \mathrm{LC}(\mathbb{R}^n)^n \to [0,\infty]$$

which is symmetric, multilinear (with respect to $\oplus$, of course) and satisfies $V(f, f, \ldots, f) = \int_{\mathbb{R}^n} f(x)dx$. We will call the number $V(f_1, f_2, \ldots f_n)$ the mixed integral of $f_1, f_2, \ldots, f_n$.

When reading the proof of our Minkowski-type theorem, one can see that log-concavity is never used in any real way. In fact, everything we said until this point will remain true, if log-concave functions are replaced with the more general quasi-concave functions:

Definition 2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called quasi-concave if

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$$

for every $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$. The class of all functions $f : \mathbb{R}^n \to [0,\infty)$ which are upper semicontinuous and quasi-concave functions will be denoted by $\mathrm{QC}(\mathbb{R}^n)$.

Similarly, a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is called quasi-convex if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

We will not have a special notation for the class of quasi-convex functions.

Quasi-concave functions are frequently used by economists (see, e.g., [11]). One of the main reasons for this is that quasi-concavity is an "ordinal property". Let us explain this point: given a function $f : \mathbb{R}^n \to [0,\infty)$ and an increasing function $\rho : [0,\infty) \to [0,\infty)$, we will say that the function $\rho \circ f$ is a rescaling of f. Many important functions in economy (e.g. the utility function) are ordinal, that is defined only up to rescaling. Remember that even if a function f is concave, its rescaling $\rho \circ f$ need not be concave. Hence one cannot talk, for example, about "concave utility functions" – concavity is not an ordinal property. In contrast, it is easy to check that if f is quasi-concave, every rescaling of it will be quasi-concave as well.

As far as we know, quasi-concave functions were never a serious object of study from the convex geometry point of view. One of the main points of this paper is to show that the realm of quasi-concave functions is the natural setting for many results and theorems.

After we define mixed integrals in section 2, section 3 is devoted to proving many different inequalities between these numbers. A sizable portion of classic convexity theory involves proving inequalities between different mixed volumes. For example, if $K \in \mathcal{K}^n$ is a convex body, its surface area is defined to be

$$S(K) = \lim_{\varepsilon \to 0^+} \frac{\mathrm{Vol}\,(K + \varepsilon D) - \mathrm{Vol}\,(K)}{\varepsilon},$$

where D is the unit Euclidean ball. It is not hard to see that $S(K)$ is a mixed volume. In fact, we have

$$S(K) = n \cdot V(\underbrace{K, K, \ldots, K}_{n-1 \text{ times}}, D).$$

The famous isoperimetric inequality states that out of all bodies with fixed volume, the Euclidean ball has the minimal surface area. More quantitatively, it is usually written as

$$S(K) \geq n \cdot \mathrm{Vol}(D)^{\frac{1}{n}} \cdot \mathrm{Vol}\,(K)^{\frac{n-1}{n}}.$$

A proof of the isoperimetric inequality, as well as all the other inequalities of mixed volumes which appear in this paper, can be found in [10].

In section 3 we discuss the question of how to prove a generalization of this theorem to quasi-concave functions. We define the surface area of a quasi-concave function f to be

$$S(f) = n \cdot V(\underbrace{f, f, \ldots, f}_{n-1 \text{ times}}, \mathbf{1}_D).$$

For log-concave functions the surface area $S(f)$ was discovered independently by Colesanti ([5]). We will give more details about his work in Example 9.

Naively, we may try and bound $S(f)$ from below in terms of the integral $\int f$. Unfortunately, we will see that no such bound can exist for arbitrary quasi-concave functions.

Instead, we will employ another approach. For every $K \in \mathcal{K}^n$ let K^* be the Euclidean ball with the same volume as K. Then the isoperimetric inequality can be stated as $S(K) \geq S(K^*)$ for every $K \in \mathcal{K}^n$. Similarly, if $f \in \mathrm{QC}\,(\mathbb{R}^n)$, we define $f^* \in \mathrm{QC}\,(\mathbb{R}^n)$ to be its symmetric decreasing rearrangement (see Definition 13, and see [7] for more information. For the purpose of this introduction we will assume that f is "nice" enough for f^* to be well-defined). We will prove the following isoperimetric inequality:

Theorem. *For every $f \in \mathrm{QC}\,(\mathbb{R}^n)$ we have $S(f) \geq S(f^*)$, with equality if and only if f is rotation invariant.*

This inequality generalizes the classic isoperimetric inequality. It can also be useful for general quasi-concave functions, because it reduces an n-dimensional problem to a 1-dimensional one – the function f^* is rotation invariant, and hence essentially "one dimensional". However, we stress again that in general, this inequality does not yield a lower bound for $S(f)$ in terms of $\int f$, as such a bound is impossible.

In section 3 we generalize many important inequalities by rewriting them as rearrangement inequalities. In particular, we extend both the Brunn-Minkowski theorem (even in its general form - see Theorem 17) and the Alexandrov-Fenchel inequality (Theorem 19). As the statements are rather involved, we will not reproduce them here. Instead, we will present an elegant corollary of Theorem 19:

Theorem. *For every* $f_1, f_2, \ldots, f_n \in \mathrm{QC}(\mathbb{R}^n)$ *we have*

$$V(f_1, f_2, \ldots, f_n) \geq V(f_1^*, f_2^*, \ldots f_n^*).$$

In section 4, we once again restrict our attention to log-concave functions. We demonstrate how one can use the results of section 3, together with a 1-dimensional analysis, to prove sharp numeric inequalities between mixed integrals. More specifically, we prove the following Alexandrov type inequality:

Theorem. *Define* $g(x) = e^{-|x|}$. *For every* $f \in \mathrm{LC}(\mathbb{R}^n)$ *with* $\max f = 1$ *and every integers* $0 \leq k < m < n$, *we have*

$$\left(\frac{W_k(f)}{W_k(g)}\right)^{\frac{1}{n-k}} \leq \left(\frac{W_m(f)}{W_m(g)}\right)^{\frac{1}{n-m}}$$

with equality if and only if $f(x) = e^{-c|x|}$ *for some* $c > 0$.

Here the numbers $W_k(f)$ are the quermassintegrals of f, defined by

$$W_k(f) = V(\underbrace{f, f, \ldots f}_{n-k \text{ times}}, \underbrace{\mathbf{1}_D, \mathbf{1}_D, \ldots, \mathbf{1}_D}_{k \text{ times}})$$

(see Example 9). In particular, the case $k = 0$, $m = 1$ gives a sharp isoperimetric inequality:

Theorem. *For every* $f \in \mathrm{LC}(\mathbb{R}^n)$ *with* $\max f = 1$ *we have*

$$S(f) \geq \left(\int f\right)^{\frac{n-1}{n}} \cdot \frac{S(g)}{\left(\int g\right)^{\frac{n-1}{n}}}$$

with equality if and only if $f(x) = e^{-c|x|}$ *for some* $c > 0$.

Finally, in section 5, we revisit the notion of rescaling. Consider for example the Brunn-Minkowski inequality: it states that for every $A, B \in \mathcal{K}^n$ we have

$$\mathrm{Vol}(A + B)^{\frac{1}{n}} \geq \mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}.$$

The most obvious way to generalize this inequality to the realm of quasi-concave functions is to ask whether

$$\left[\int (f \oplus g)\right]^{\frac{1}{n}} \geq \left[\int f\right]^{\frac{1}{n}} + \left[\int g\right]^{\frac{1}{n}}$$

for arbitrary functions $f, g \in \mathrm{QC}(\mathbb{R}^n)$. Unfortunately, this is false, just like the most naive way to generalize the isoperimetric inequality turned out to be false. One way we already discussed to remedy the situation is to reinterpret the Brunn-Minkowski inequality as a rearrangement inequality. The resulting inequality will read

$$(f \oplus g)^* \geq f^* \oplus g^*$$

(see Proposition 15 for an exact statement and a detailed discussion). However, there is also a second way. Remember that quasi-concave functions are often ordinal functions, i.e. functions defined only up to rescaling. In such a case, we can ask a more delicate question: Is it possible to choose rescalings of f and g in such a way that the Brunn-Minkowski inequality will hold? Often, the answer is "yes":

Theorem. *Assume f and g are "sufficiently nice" quasi-concave functions. Then one can rescale f to a function $\tilde{f}$ in such a way that*

$$\left[\int\left(\tilde{f}\oplus g\right)\right]^{\frac{1}{n}}\geq\left[\int\tilde{f}\right]^{\frac{1}{n}}+\left[\int g\right]^{\frac{1}{n}}.$$

Of course, the exact definition of "sufficiently nice" will be given in section 5. Section 5 also contains a "rescaled" version of the Alexandrov-Fenchel inequality, which we will not produce here.

2. Minkowski theorem for quasi-concave functions

The main goal of this section is to establish the various properties of the addition $\oplus$ from Definition 1, including a Minkowski type theorem. Remember that the sum $f\oplus g$ of two quasi-concave functions is defined by

$$\left(f\oplus g\right)(x)=\sup_{y\in\mathbb{R}^n}\min\left\{f(y),g(x-y)\right\}.$$

The first thing we do is give an alternative, more intuitive definition for $\oplus$. We start by defining

Definition 3. For a quasi-convex function $\varphi:\mathbb{R}^n\to\mathbb{R}$ and $t\in\mathbb{R}$, we define

$$\underline{K}_t(\varphi)=\left\{x\in\mathbb{R}^n:\ \varphi(x)\leq t\right\}.$$

Similarly for $f\in\mathrm{QC}\left(\mathbb{R}^n\right)$ and $t>0$, we define

$$\overline{K}_t(f)=\left\{x\in\mathbb{R}^n:\ f(x)\geq t\right\}.$$

Since φ is quasi-convex, its lower level sets $\underline{K}_t(\varphi)$ are convex. If it also lower semicontinuous, the sets $\underline{K}_t(\varphi)$ are closed as well. Therefore $\underline{K}_t(\varphi)\in\mathcal{K}^n$, and for the similar reasons $\overline{K}_t(f)\in\mathcal{K}^n$ for $f\in\mathrm{QC}\left(\mathbb{R}^n\right)$. We can now explain our addition in terms of level sets:

Proposition 4. *Assume $f,g\in\mathrm{QC}\left(\mathbb{R}^n\right)$ and the level sets $\overline{K}_t(f)$ are compact for all $t>0$. Then we have*

$$\overline{K}_t(f\oplus g)=\overline{K}_t(f)+\overline{K}_t(g).$$

Similarly, if $\lambda>0$, and with no compactness assumption, we get

$$\overline{K}_t(\lambda\odot f)=\lambda\cdot\overline{K}_t(f).$$

For lower-semicontinuous quasi-convex functions we have a similar result, with the lower level sets $\underline{K}_t(\varphi)$ playing the role of the upper level sets $\overline{K}_t(f)$.

Proof. If $y_0\in\overline{K}_t(f)$ and $z_0\in\overline{K}_t(g)$ then

$$\begin{aligned}(f\oplus g)(y_0+z_0) &= \sup_{y\in\mathbb{R}^n}\min\left\{f(y),g(y_0+z_0-y)\right\}\\ &\geq \min\left\{f(y_0),g(y_0+z_0-y_0)\right\})\\ &\geq \min\left\{t,t\right\}=t,\end{aligned}$$

so $y_0 + z_0 \in \overline{K}_t(f+g)$, and it follows that

$$\overline{K}_t(f \oplus g) \supseteq \overline{K}_t(f) + \overline{K}_t(g).$$

For the other inclusion, fix $t > 0$ and assume $x_0 \in \overline{K}_t(f+g)$. This means that

$$(f \oplus g)(x_0) = \sup_{y \in \mathbb{R}^n} \min\{f(y), g(x_0 - y)\} \geq t,$$

so we can choose a sequence $\{y_m\}_{m=1}^{\infty}$ such that $f(y_m) > t - \frac{1}{m}$ and $g(x_0 - y_m) > t - \frac{1}{m}$. Fix m_0 large enough that $t - \frac{1}{m_0} > 0$. For every $m \geq m_0$ we have $y_m \in \overline{K}_{t-\frac{1}{m_0}}(f)$, and since $\overline{K}_{t-\frac{1}{m_0}}(f)$ is compact we can assume without loss of generality that $y_m \to y$ as $m \to \infty$. By the upper semi-continuity of f and g we get

$$\begin{aligned} f(y) &\geq \limsup_{m\to\infty} f(y_m) \geq \limsup_{m\to\infty} t - \frac{1}{m} = t \\ g(x_0 - y) &\geq \limsup_{m\to\infty} g(x_0 - y_m) \geq \limsup_{m\to\infty} t - \frac{1}{m} = t, \end{aligned}$$

and then $x_0 = y + (x_0 - y) \in \overline{K}_t(f) + \overline{K}_t(g)$. This proves the first assertion of the proposition. The assertion $\overline{K}_t(\lambda \odot f) = \lambda \cdot \overline{K}_t(f)$ is trivial. □

The compactness assumption in Proposition 4 is necessary. To see this, define $f, g \in \mathrm{QC}(\mathbb{R})$ by $f(x) = \frac{\pi}{2} + \arctan x$ and $g(x) = \frac{\pi}{2} - \arctan x$. Then

$$(f \oplus g)(x) = \sup_{y \in \mathbb{R}^n} \min\left\{\frac{\pi}{2} + \arctan y, \frac{\pi}{2} + \arctan(y - x)\right\} = \pi$$

for all x, and we have

$$\overline{K}_\pi(f) + \overline{K}_\pi(g) = \emptyset + \emptyset = \emptyset \neq \mathbb{R} = \overline{K}_\pi(f \oplus g).$$

This is a minor detail, however, as our main interest is in functions $f \in \mathrm{QC}(\mathbb{R}^n)$ such that $0 < \int f < \infty$, and for such functions the level sets $\overline{K}_t(f)$ are indeed compact. It is also easy to check and useful to notice that even without any compactness assumptions, we still have

$$\{x : (f \oplus g)(x) > t\} \subseteq \overline{K}_t(f) + \overline{K}_t(g).$$

We will now prove that the sum of convex functions is indeed convex:

Proposition 5. *For $\varphi, \psi \in \mathrm{Cvx}(\mathbb{R}^n)$ the function $\varphi \oplus \psi$ is convex. If, in addition, the sets $\underline{K}_t(\varphi)$ are compact, then $\varphi \oplus \psi$ is lower semicontinuous, so $\varphi \oplus \psi \in \mathrm{Cvx}(\mathbb{R}^n)$.*

Proof. We will verify directly that $\varphi \oplus \psi$ is convex. Fix $x_0, x_1 \in \mathbb{R}^n$ and $0 < \lambda < 1$. For every $\varepsilon > 0$ we can find $y_0, y_1 \in \mathbb{R}^n$ such that

$$\begin{aligned} (\varphi \oplus \psi)(x_0) &= \inf_{y \in \mathbb{R}^n} \max\{\varphi(y), \psi(x_0 - y)\} \geq \max\{\varphi(y_0), \psi(x_0 - y_0)\} - \varepsilon \\ (\varphi \oplus \psi)(x_1) &= \inf_{y \in \mathbb{R}^n} \max\{\varphi(y), \psi(x_1 - y)\} \geq \max\{\varphi(y_1), \psi(x_1 - y_1)\} - \varepsilon. \end{aligned}$$

Define $x_\lambda = (1-\lambda)x_0 + \lambda x_1$ and $y_\lambda = (1-\lambda)y_0 + \lambda y_1$. Notice that

$$\begin{aligned}
\varphi(y_\lambda) &\leq (1-\lambda)\varphi(y_0) + \lambda\varphi(y_1) \\
&\leq (1-\lambda)\max\{\varphi(y_0), \psi(x_0-y_0)\} + \lambda\max\{\varphi(y_1), \psi(x_1-y_1)\} \\
&\leq (1-\lambda)\left[(\varphi\oplus\psi)(x_0) + \varepsilon\right] + \lambda\left[(\varphi\oplus\psi)(x_1) + \varepsilon\right] \\
&= (1-\lambda)\left[(\varphi\oplus\psi)(x_0)\right] + \lambda\left[(\varphi\oplus\psi)(x_1)\right] + \varepsilon,
\end{aligned}$$

and similarly

$$\begin{aligned}
\psi(x_\lambda - y_\lambda) &\leq (1-\lambda)\psi(x_0-y_0) + \lambda\psi(x_1-y_1) \\
&\leq (1-\lambda)\left[(\varphi\oplus\psi)(x_0)\right] + \lambda\left[(\varphi\oplus\psi)(x_1)\right] + \varepsilon.
\end{aligned}$$

Therefore

$$\begin{aligned}
(\varphi\oplus\psi)(x_\lambda) &= \inf_{y\in\mathbb{R}^n}\max\{\varphi(y), \psi(x_\lambda - y)\} \leq \max\{\varphi(y_\lambda), \psi(x_\lambda - y_\lambda)\} \\
&\leq (1-\lambda)\left[(\varphi\oplus\psi)(x_0)\right] + \lambda\left[(\varphi\oplus\psi)(x_1)\right] + \varepsilon.
\end{aligned}$$

Taking $\varepsilon \to 0$, we obtain the result.

If the level sets $\underline{K}_t(\varphi)$ are compact, one may apply Proposition 4 and conclude that for every $t \in \mathbb{R}$

$$\underline{K}_t(\varphi\oplus\psi) = \underline{K}_t(\varphi) + \underline{K}_t(\psi).$$

Since $\underline{K}_t(\varphi)$ is compact and $\underline{K}_t(\psi)$ is closed, their Minkowski sum $\underline{K}_t(\varphi\oplus\psi)$ is closed as well. This implies that $\varphi\oplus\psi$ is lower semicontinuous, and the proof is complete. □

Again, the compactness assumption is necessary. This is not a surprise, because it is well known that even the corresponding theorem for convex bodies fails without compactness: Define

$$\begin{aligned}
K_1 &= \left\{(x,y)\in\mathbb{R}^2 :\ x>0,\ y\geq \frac{1}{x}\right\} \\
K_2 &= \left\{(x,y)\in\mathbb{R}^2 :\ x>0,\ y\leq -\frac{1}{x}\right\}.
\end{aligned}$$

Then K_1 and K_2 are closed, convex sets, but their sum $K_1+K_2 = \left\{(x,y)\in\mathbb{R}^2 :\ x>0\right\}$ is not closed. If we now define

$$\varphi_i(x,y) = \mathbf{1}_{K_i}^\infty(x,y) = \begin{cases} 0 & (x,y)\in K_i \\ \infty & \text{otherwise,}\end{cases}$$

then each φ_i is lower semicontinuous, but $\varphi_1\oplus\varphi_2 = \mathbf{1}_{K_1+K_2}^\infty$ is not.

We are now ready to prove Minkowski's theorem for our addition:

Theorem 6. *Fix $f_1, f_2, \ldots, f_m \in \mathrm{QC}(\mathbb{R}^n)$. Then the function $F : (\mathbb{R}^+)^m \to [0,\infty]$, defined by*

$$F(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) = \int \left[(\varepsilon_1\odot f_1)\oplus(\varepsilon_2\odot f_2)\oplus\cdots\oplus(\varepsilon_m\odot f_m)\right]$$

is a homogenous polynomial of degree n, with non-negative coefficients. If we write

$$F(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) = \sum_{i_1,i_2,\ldots,i_n=1}^{m} \varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_n}\cdot V(f_{i_1}, f_{i_2}\ldots, f_{i_n})$$

for a symmetric function V, then

$$V(f_1, f_2, \ldots, f_n) = \int_0^\infty V\left(\overline{K}_t(f_1), \overline{K}_t(f_2), \ldots, \overline{K}_t(f_n)\right) dt.$$

Proof. Define $h = (\varepsilon_1 \odot f_1) \oplus (\varepsilon_2 \odot f_2) \oplus \cdots \oplus (\varepsilon_m \odot f_m)$. Using Fubini's theorem we can integrate by level sets and obtain

$$\int h = \int_0^\infty \left|\overline{K}_t(h)\right| dt,$$

where $|\cdot|$ denotes the Lebesgue volume.

Applying Proposition 4, we see that if the upper level sets $\overline{K}_t(f_i)$ are all compact then

$$\overline{K}_t(h) = \varepsilon_1 \overline{K}_t(f_1) + \varepsilon_2 \overline{K}_t(f_2) + \cdots + \varepsilon_m \overline{K}_t(f_m),$$

so we can integrate and obtain

$$\int_0^\infty \left|\overline{K}_t(h)\right| dt = \int_0^\infty \left|\varepsilon_1 \underline{K}_t(f_1) + \varepsilon_2 \underline{K}_t(f_2) + \cdots + \varepsilon_m \underline{K}_t(f_m)\right| dt.$$

In fact, by being a bit careful, it is possible to obtain the above formula even without assuming compactness. Indeed, by the discussion after Proposition 4, we see that we always have

$$\overline{K}_t(h) \supseteq \varepsilon_1 \overline{K}_t(f_1) + \varepsilon_2 \overline{K}_t(f_2) + \cdots + \varepsilon_m \overline{K}_t(f_m) \supseteq \{x : \ h(x) > t\}.$$

Since $\left|\overline{K}_t(h)\right| = |\{x : \ h(x) > t\}|$ for all but countably many values of t, we get that

$$\left|\overline{K}_t(h)\right| = \left|\varepsilon_1 \overline{K}_t(f_1) + \varepsilon_2 \overline{K}_t(f_2) + \cdots + \varepsilon_m \overline{K}_t(f_m)\right|$$

for all but countably many values of t, so we can still integrate and get the formula we want.

Now we apply the classic Minkowski theorem and obtain

$$\begin{aligned}
\int h &= \int_0^\infty \left|\varepsilon_1 \underline{K}_t(f_1) + \varepsilon_2 \underline{K}_t(f_2) + \cdots + \varepsilon_m \underline{K}_t(f_m)\right| dt \\
&= \int_0^\infty \sum_{i_1, i_2, \ldots, i_n = 1}^{m} \varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_n} \cdot V\left(\overline{K}_t(f_{i_1}), \overline{K}_t(f_{i_2}), \ldots, \overline{K}_t(f_{i_m})\right) dt \\
&= \sum_{i_1, i_2, \ldots, i_n = 1}^{m} \varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_n} \cdot \int_0^\infty V\left(\overline{K}_t(f_{i_1}), \overline{K}_t(f_{i_2}), \ldots, \overline{K}_t(f_{i_m})\right) dt
\end{aligned}$$

which is exactly what we wanted. □

From Theorem 6 the following definition becomes natural:

Definition 7. If $f_1, \ldots, f_n \in \mathrm{QC}(\mathbb{R}^n)$ we define their mixed integral as

$$V(f_1, f_2, \ldots, f_n) = \int_0^\infty V\left(\overline{K}_t(f_1), \overline{K}_t(f_2), \ldots, \overline{K}_t(f_n)\right) dt.$$

The mixed integrals $V(f_1, f_2, \ldots, f_n)$ are exactly the polarization we were looking for – it is a symmetric, multilinear functional such that $V(f, f, \ldots, f) = \int f$ for all $f \in \mathrm{LC}(\mathbb{R}^n)$.

We will now give a couple of examples of mixed integrals:

Example 8. One can view mixed volumes as a special case of mixed integrals. Fix bodies $K_1, \ldots, K_n$ and a parameter $p > 0$, and define $f_i(x) = \exp\left(-\|x\|_{K_i}^p\right) \in \mathrm{QC}\left(\mathbb{R}^n\right)$. Then for every $0 < t < 1$ we have

$$\overline{K}_t(f_i) = \left\{x : \ \exp\left(-\|x\|_{K_i}^p\right) \geq t\right\} = \left\{x : \ \|x\|_{K_i} \leq \left(\log \frac{1}{t}\right)^{\frac{1}{p}}\right\} = \left(\log \frac{1}{t}\right)^{\frac{1}{p}} \cdot K_i.$$

Hence,

$$\begin{aligned} V(f_1, \ldots, f_n) &= \int_0^1 V\left(\left(\log \frac{1}{t}\right)^{\frac{1}{p}} K_1, \ldots, \left(\log \frac{1}{t}\right)^{\frac{1}{p}} K_n\right) dt \\ &= \left[\int_0^1 \left(\log \frac{1}{t}\right)^{\frac{n}{p}} dt\right] \cdot V(K_1, \ldots, K_n), \end{aligned}$$

so the mixed integral of $f_1, \ldots, f_n$ is the same as the mixed volume of $K_1, \ldots, K_n$, up to normalization. In particular, taking $p \to \infty$ we get

$$V\left(\mathbf{1}_{K_1}, \mathbf{1}_{K_2}, \ldots, \mathbf{1}_{K_n}\right) = V\left(K_1, \ldots, K_n\right),$$

which can also be seen directly from the definition.

Of course, one can make this example even more general, by choosing f_i to be any integrable function such that $\overline{K}_t(f_i)$ is always homothetic to K_i.

Example 9. Fix any $f \in \mathrm{QC}\left(\mathbb{R}^n\right)$, and assume for simplicity that $f(x) \leq 1$ for all x. Also, fix a convex set $K \in \mathcal{K}^n$. It is not hard to see that for every $\varepsilon > 0$ we have

$$f \oplus (\varepsilon \odot \mathbf{1}_K) = f \star (\varepsilon \cdot \mathbf{1}_K) = \sup_{y \in \varepsilon K} f(x - y).$$

Here $\star$ is the sup-convolution operation described in the introduction, and $\cdot$ is the induced homothety operation defined by $(\lambda \cdot g)(x) = g\left(\frac{x}{\lambda}\right)^{\lambda}$. It follows from Theorem 6 that the integral $\int f \star (\varepsilon \cdot \mathbf{1}_K)$ is a polynomial in ε. In other words, in this restricted case, we obtain a Minkowski type theorem for the sup-convolution operation.

In particular, define the ε-extension of f to be

$$f_\varepsilon(x) = \sup_{|y| \leq \varepsilon} f(x + y).$$

Since $f_\varepsilon = f \oplus (\varepsilon \odot \mathbf{1}_D)$, where D is the Euclidean ball, the integral $\int f_\varepsilon$ is a polynomial in ε. Its coefficients,

$$W_k(f) = V(\underbrace{f, f, \ldots f}_{n-k \text{ times}}, \underbrace{\mathbf{1}_D, \mathbf{1}_D, \ldots, \mathbf{1}_D}_{k \text{ times}})$$

will be called the quermassintegrals of f. This is in analogy to the classic quermassintegrals, which are defined for every $K \in \mathcal{K}^n$ by

$$W_k(K) = V(\underbrace{K, K, \ldots K}_{n-k \text{ times}}, \underbrace{D, D, \ldots, D}_{k \text{ times}}).$$

This example was discovered independently by Colesanti [5], who also proved that these numbers share several important properties of the classic quermassintegrals.

Of course, there is no reason to use only one convex body. One easily checks that for every $f \in \mathrm{QC}(\mathbb{R}^n)$ and every convex sets $K_1, K_2, \ldots, K_m \in \mathcal{K}^n$, the integral

$$\int \left[f \star (\varepsilon_1 \odot \mathbf{1}_{K_1}) \star (\varepsilon_2 \odot \mathbf{1}_{K_2}) \star \cdots \star (\varepsilon_m \odot \mathbf{1}_{K_m})\right]$$

is a polynomial in $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$.

We see that the operation $\oplus$ satisfies a Minkowski theorem, while the standard inf-convolution $\Box$ does not. It is interesting to notice that, nonetheless, the operations $\oplus$ and $\Box$ are not so different. In fact, for every positive convex functions φ and ψ we have

$$\frac{1}{2}\left(\varphi \Box \psi\right)(x) \le \left(\varphi \oplus \psi\right)(x) \le \left(\varphi \Box \psi\right)(x).$$

This follows from the following proposition:

Proposition 10. *Fix convex functions $\varphi_1, \varphi_2, \ldots \varphi_k \in \mathrm{Cvx}(\mathbb{R}^n)$ such that $\varphi_i(x) \ge 0$ for all $1 \le i \le k$ and $x \in \mathbb{R}^n$. For $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$, define*

$$\begin{aligned} g_1(x) &= \left(\lambda_1 \cdot \varphi_1 \Box \lambda_2 \cdot \varphi_2 \Box \cdots \Box \lambda_k \cdot \varphi_k\right)(x) \\ g_2(x) &= \left(\lambda_1 \odot \varphi_1 \oplus \lambda_2 \odot \varphi_2 \oplus \cdots \oplus \lambda_k \odot \varphi_k\right)(x) \end{aligned}$$

Then

$$\left(\sum_i \lambda_i\right)^{-1} g_1(x) \le g_2(x) \le \left(\min \lambda_i\right)^{-1} g_1(x).$$

Proof. By definition we have

$$\begin{aligned} g_1(x) &= \inf_{\sum \lambda_i y_i = x} \sum_{i=1}^{k} \lambda_i f_i(y_i), \\ g_2(x) &= \inf_{\sum \lambda_i y_i = x} \max_{1 \le i \le k} \{f_i(y_i)\}. \end{aligned}$$

For every non-negative numbers $a_1, \ldots, a_k$ and positive coefficients $\lambda_1, \ldots, \lambda_k$ we have

$$\frac{\sum_i \lambda_i a_i}{\sum_i \lambda_i} \le \max_i a_i \le \frac{\sum_i \lambda_i a_i}{\min \lambda_i},$$

and the result follows immediately. $\square$

To conclude this section, let us digress slightly and discuss a relation between the above constructions and the notion of polarity. From Proposition 4 we see that our sum $\oplus$ operates on level sets: In order to find $\underline{K}_t(\varphi \oplus \psi)$ it is enough to know $\underline{K}_t(\varphi)$ and $\underline{K}_t(\psi)$. Another important structure in convexity is that of polarity. Let $\varphi : \mathbb{R}^n \to [0, \infty]$ be a lower semicontinuous convex function with $\varphi(0) = 0$ – such functions are called geometric convex functions. The polar function of φ, which we will denote by φ°, is defined using the so-called $\mathcal{A}$-transform of φ,

$$\varphi^\circ(x) := \left(\mathcal{A}\varphi\right)(x) = \sup_{y \in \mathbb{R}^n} \frac{\langle x, y\rangle - 1}{\varphi(y)}$$

(see [1] for a detailed discussion. See also [9] section 15, as well as [8] for historical remarks). Polarity does not work on level sets. However, it is interesting to notice that it is "almost" the case:

Proposition 11. *For every geometric convex function φ and every $t > 0$ we have*

$$\underline{K}_{1/t}(\varphi)^\circ \subseteq \underline{K}_t(\varphi^\circ) \subseteq 2\underline{K}_{1/t}(\varphi)^\circ.$$

In an informal way, one can say that the polars of the level sets are "almost" the level sets of the polar.

Proof. The easier inclusion is the right one. Assume $x \in \underline{K}_t(\varphi^\circ)$ and take any $y \in \underline{K}_{1/t}(\varphi)$. We have

$$t \geq \varphi^\circ(x) \geq \frac{\langle x, y\rangle - 1}{\varphi(y)} \geq \frac{\langle x, y\rangle - 1}{1/t} = t\left(\langle x, y\rangle - 1\right),$$

and when we divide by t we get $\langle x, y\rangle - 1 \leq 1$, which implies $\left\langle \frac{x}{2}, y\right\rangle \leq 1$. Since $y \in \underline{K}_{1/t}(\varphi)$ was arbitrary it follows that $\frac{x}{2} \in \underline{K}_{1/t}(\varphi)^\circ$, so $x \in 2\underline{K}_{1/t}(\varphi)^\circ$.

For the left inclusion, fix $x \in \underline{K}_{1/t}(\varphi)^\circ$ and take any $y \in \mathbb{R}^n$. If $\varphi(y) \leq \frac{1}{t}$ then $y \in \underline{K}_{1/t}(\varphi)$, so $\langle x, y\rangle \leq 1$ and

$$\frac{\langle x, y\rangle - 1}{\varphi(y)} \leq 0 \leq t.$$

If, on the other hand, $\varphi(y) = s > \frac{1}{t}$, then $st > 1$, and using the convexity of φ we get

$$\varphi\left(\frac{y}{st}\right) = \varphi\left(\frac{1}{st}\cdot y + \left(1 - \frac{1}{st}\right)\cdot 0\right) \leq \frac{1}{st}\varphi(y) + \left(1 - \frac{1}{st}\right)\varphi(0) = \frac{1}{t}.$$

Hence $\frac{y}{st} \in L_{1/t}(\varphi)$, so $\left\langle x, \frac{y}{st}\right\rangle \leq 1$, and then

$$\frac{\langle x, y\rangle - 1}{\varphi(y)} \leq \frac{st - 1}{s} = t - \frac{1}{s} \leq t.$$

All together we get

$$\varphi^\circ(x) = \sup_{y\in\mathbb{R}^n} \frac{\langle x, y\rangle - 1}{\varphi(y)} \leq t,$$

so $x \in \underline{K}_t(\varphi^\circ)$, like we wanted. □

Assume φ is any function which is geometric, quasi-convex and lower semicontinuous. We define its dual function φ^* via the relation

$$\underline{K}_t(\varphi^*) = \underline{K}_{1/t}(\varphi)^\circ$$

for any $t > 0$. It is easy to see that φ^* is also geometric, quasi-convex and lower semicontinuous. Furthermore, we have $(\varphi^*)^* = \varphi$, so the operation $*$ really defines a duality relation on quasi-convex functions. Proposition 11 tells us that for convex functions, the operations $\circ$ and $*$ are very similar.

3. Rearrangement inequalities

In this section we will generalize several classic inequalities concerning mixed volumes to the realm of quasi-concave functions. For simplicity, we will always assume our quasi-concave functions are geometric:

Definition 12. A function $f \in \mathrm{QC}(\mathbb{R}^n)$ is called geometric if

$$\max_{x\in\mathbb{R}^n} f(x) = f(0) = 1.$$

The class of all geometric quasi-concave functions will be denoted by $\mathrm{QC}_0(\mathbb{R}^n)$. We define the class $\mathrm{LC}_0(\mathbb{R}^n)$ of geometric log-concave functions in a similar way.

In section 4, the fact that the functions involved are geometric will play a crucial role. Here, however, this is merely a matter of convenience, allowing us to ignore some technical details – many of the results will remain true even without this assumption.

Remember from the introduction that our first goal is to state and prove an extension of the isoperimetric inequality to our case: we want to give a lower bound on

$$S(f) = n \cdot W_1(f)$$

in terms of the integral $\int f$ (the notation W_k appeared in Example 9). Unfortunately, this is impossible: In Remark 28 we will construct a sequence $f_k \in \mathrm{QC}_0\left(\mathbb{R}^2\right)$ with $\int f_k = 1$, but $S(f_k) \stackrel{k\to\infty}{\longrightarrow} 0$. Hence we follow another route and define:

Definition 13. (i) For a compact $K \in \mathcal{K}^n$, define

$$K^* = \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(D)}\right)^{\frac{1}{n}} D.$$

In other words, K^* is the Euclidean ball with the same volume as K.

(ii) For $f \in \mathrm{QC}_0(\mathbb{R}^n)$ with compact upper level sets, define its symmetric decreasing rearrangement f^* using the relation

$$\overline{K}_t(f^*) = \overline{K}_t(f)^*.$$

It is easy to see that this definition really defines a unique function $f^* \in \mathrm{QC}_0(\mathbb{R}^n)$, which is rotation invariant.

Since K^* is a ball with the same volume as K, the isoperimetric inequality tells us that $S(K) \geq S(K^*)$ for every convex body $K \in \mathcal{K}^n$. This means we can think about the isoperimetric inequality as a rearrangement inequality, and this point of view can be extended to quasi-concave functions:

Proposition 14. *If $f \in \mathrm{QC}_0(\mathbb{R}^n)$ has compact level sets, then $S(f) \geq S(f^*)$, with equality if and only if f is rotation invariant.*

Proof. Notice that for every function $g \in \mathrm{QC}_0(\mathbb{R}^n)$ we have

$$\begin{aligned} S(g) &= n \cdot W_1(g) = n \cdot V(\underbrace{g, g, \ldots, g}_{n-1 \text{ times}}, \mathbf{1}_D) \\ &= \int_0^1 n \cdot V\left(\overline{K}_t(g), \overline{K}_t(g), \ldots, \overline{K}_t(g), D\right) dt = \int_0^1 S(\overline{K}_t(g)) dt. \end{aligned}$$

Using the classic isoperimetric inequality we get

$$S(f^*) = \int_0^1 S(\overline{K}_t(f^*))dt = \int_0^1 S(\overline{K}_t(f)^*)dt \le \int_0^1 S(\overline{K}_t(f))dt = S(f),$$

which is what we wanted.

If $S(f^*) = S(f)$ then $S(\overline{K}_t(f)^*) = S(\overline{K}_t(f))$ for all t. Again by the classic isoperimetric inequality this implies that $\overline{K}_t(f)$ is always a ball, so f is rotation invariant. □

In classic convexity, the isoperimetric inequality follows easily from the Brunn-Minkowski theorem, which states that for any (say convex) bodies $A, B \in \mathcal{K}^n$, we have

$$\mathrm{Vol}(A+B)^{\frac{1}{n}} \ge \mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}.$$

The Brunn-Minkowski theorem can also be written as a rearrangement inequality: $(A+B)^* \supseteq A^* + B^*$. The corresponding result for quasi-concave functions is

Proposition 15. *If $f, g \in \mathrm{QC}_0(\mathbb{R}^n)$ have compact level sets, then $(f \oplus g)^* \ge f^* \oplus g^*$.*

In particular, we have

$$\int f \oplus g = \int (f \oplus g)^* \ge \int f^* \oplus g^*.$$

Proof. It is enough to prove that $\overline{K}_t\left((f \oplus g)^*\right) \supseteq \overline{K}_t\left(f^* \oplus g^*\right)$ for all t . But

$$\begin{aligned}
\overline{K}_t\left((f \oplus g)^*\right) &= \overline{K}_t\left(f \oplus g\right)^* = \left(\overline{K}_t(f) + \overline{K}_t(g)\right)^* \\
&\supseteq \overline{K}_t(f)^* + \overline{K}_t(g)^* = \overline{K}_t(f^*) + \overline{K}_t(g^*) \\
&= \overline{K}_t\left(f^* \oplus g^*\right),
\end{aligned}$$

so the result holds. □

We would now like to take even more general inequalities and cast them to our setting. For example, the most general Brunn-Minkowski inequality for mixed volumes states that for every $A, B, K_1, K_2, \ldots, K_{n-m} \in \mathcal{K}^n$ we have

$$\begin{aligned}
V(\underbrace{A+B, \ldots, A+B}_{m \text{ times}}, K_1, \ldots, K_{n-m})^{\frac{1}{m}} \quad &\ge \quad V(A, \ldots, A, K_1, \ldots, K_{n-m})^{\frac{1}{m}} + \\
&\quad + V(B, \ldots, B, K_1, \ldots, K_{n-m})^{\frac{1}{m}}.
\end{aligned}$$

In order to write such an inequality in our language, we will need to define a generalized concept of rearrangement:

Definition 16. (i) A size functional is a function $\Phi : \mathcal{K}^n \to [0, \infty]$ of the form

$$\Phi(A) = V(\underbrace{A, \ldots, A}_{m \text{ times}}, K_1, \ldots, K_{n-m}),$$

for fixed compact bodies $K_1, K_2, \ldots, K_{n-m}$ with non-empty interior. We will say that Φ is of degree m.

(ii) If $K \in \mathcal{K}^n$ is compact and Φ is a size functional of degree m, define

$$K^{\Phi} = \left(\frac{\Phi(K)}{\Phi(D)}\right)^{\frac{1}{m}} \cdot D.$$

If, in addition, $f \in \mathrm{QC}_0(\mathbb{R}^n)$ has compact level sets, define $f^{\Phi} \in \mathrm{QC}_0(\mathbb{R}^n)$ by the relation

$$\overline{K}_t\left(f^{\Phi}\right) = \overline{K}_t(f)^{\Phi}.$$

In particular, we have $K^{\mathrm{Vol}} = K^*$ and $f^{\mathrm{Vol}} = f^*$ for a convex body K and a quasi-concave function f. Notice that K^{Φ} is the Euclidean ball of the same "size" as K, where size is defined using the functional Φ.

For functions the intuition is similar. If $\Phi : \mathcal{K}^n \to [0,\infty]$ is a size functional, we can extend the domain of Φ to all of $\mathrm{QC}_0(\mathbb{R}^n)$ in a natural way: If

$$\Phi(A) = V(\underbrace{A, \ldots, A}_{m \text{ times}}, K_1, \ldots, K_{n-m}),$$

then

$$\Phi(f) = V(\underbrace{f, \ldots, f}_{m \text{ times}}, \mathbf{1}_{K_1}, \ldots, \mathbf{1}_{K_{n-m}}).$$

Notice that for every quasi-concave function $g \in \mathrm{QC}_0(\mathbb{R}^n)$ we have

$$\Phi(g) = \int_0^1 \Phi\left(\overline{K}_t(g)\right) dt,$$

so in particular

$$\begin{aligned} \Phi(f^{\Phi}) &= \int_0^1 \Phi\left(\overline{K}_t(f^{\Phi})\right) dt = \int_0^1 \Phi\left(\overline{K}_t(f)^{\Phi}\right) dt \\ &= \int_0^1 \Phi\left(\overline{K}_t(f)\right) dt = \Phi(f). \end{aligned}$$

This means that f^{Φ} is a rotation invariant function with the same "size" as f.

Now we can write the general Brunn-Minkowski inequality as a generalized rearrangement inequality, both for convex bodies and for convex functions:

Theorem 17. *Let Φ be a size functional. Then*

(i) $(A+B)^{\Phi} \supseteq A^{\Phi} + B^{\Phi}$ *for every compact convex bodies* $A, B \in \mathcal{K}^n$.

(ii) $(f \oplus g)^{\Phi} \geq f^{\Phi} \oplus g^{\Phi}$ *for every quasi-concave functions* $f, g \in \mathrm{QC}_0(\mathbb{R}^n)$ *with compact level sets.*

Proof. First we deal with the case of bodies, where the proposition is just a restatement of the generalized Brunn-Minkowski inequality: Notice that $(A+B)^{\Phi}$ is a ball of radius

$$\left(\frac{\Phi(A+B)}{\Phi(D)}\right)^{\frac{1}{m}},$$

where m is the degree of Φ. Similarly, $A^{\Phi} + B^{\Phi}$ is a ball of radius

$$\left(\frac{\Phi(A)}{\Phi(D)}\right)^{\frac{1}{m}} + \left(\frac{\Phi(B)}{\Phi(D)}\right)^{\frac{1}{m}},$$

so the result follows immediately.

For quasi-concave functions, we argue just like in Proposition 15:

$$\begin{aligned}\overline{K}_t\left((f\oplus g)^{\Phi}\right) &= \overline{K}_t\left(f\oplus g\right)^{\Phi} = \left(\overline{K}_t(f)+\overline{K}_t(g)\right)^{\Phi} \supseteq \overline{K}_t(f)^{\Phi}+\overline{K}_t(g)^{\Phi}\\ &= \overline{K}_t(f^{\Phi})+\overline{K}_t(g^{\Phi}) = \overline{K}_t\left(f^{\Phi}\oplus g^{\Phi}\right),\end{aligned}$$

so $(f\oplus g)^{\Phi} \geq f^{\Phi}+g^{\Phi}$ like we wanted. □

Again, we see as a corollary that

$$\Phi(f\oplus g) = \Phi\left((f\oplus g)^{\Phi}\right) \geq \Phi\left(f^{\Phi}\oplus g^{\Phi}\right).$$

Other geometric inequalities can be written in the same form as well. The Alexandrov inequalities for quermassintegrals state that for every $K\in\mathcal{K}^n$ and every $0\leq i<j<n$ we have

$$\left(\frac{W_j(K)}{W_j(D)}\right)^{\frac{1}{n-j}} \geq \left(\frac{W_i(K)}{W_i(D)}\right)^{\frac{1}{n-i}},$$

with equality if and only if K is a ball. In the language of rearrangements, we can write:

Proposition 18. *Fix $0\leq i<j<n$. Then*

(i) $K^{W_j}\supseteq K^{W_i}$ *for every compact, convex body $K\in\mathcal{K}^n$. If $K^{W_j}=K^{W_i}$ then K is a ball.*
(ii) $f^{W_j}\geq f^{W_i}$ *for every quasi-concave $f\in\mathrm{QC}_0(\mathbb{R}^n)$ with compact level sets. If $f^{W_j}=f^{W_i}$, then f is rotation invariant.*

Proof. Since K^{W_j} is simply a ball of radius

$$\left(\frac{W_j(K)}{W_j(D)}\right)^{\frac{1}{n-j}},$$

the first claim is just a reformulation of the Alexandrov inequalities. The second claim will follow easily by comparing level sets:

$$\overline{K}_t(f^{W_j}) = \overline{K}_t(f)^{W_j} \supseteq \overline{K}_t(f)^{W_i} = \overline{K}(f^{W_i}).$$

□

Again, we see as a corollary that if $f\in\mathrm{QC}_0(\mathbb{R}^n)$ then for every $j>i$ the function $g=f^{W_i}$ is rotation invariant, and satisfies

$$\begin{aligned}W_i(g) &= W_i(f^{W_i}) = W_i(f)\\ W_j(g) &= W_j(f^{W_i}) \leq W_j(f^{W_j}) = W_j(f).\end{aligned}$$

The case $i=0$, $j=1$ is just the isoperimetric inequality proven earlier.

Now we would like to prove a version of the powerful Alexandrov-Fenchel inequality. We will state and prove the proposition, and the proof will explain in what way this is really an Alexandrov-Fenchel type theorem

Theorem 19. *Fix a size functional*

$$\Phi(A) = V(\underbrace{A, \ldots, A}_{m \text{ times}}, K_1, \ldots, K_{n-m}).$$

Then:

(i) For every compact bodies $A_1, \ldots, A_m \in \mathcal{K}^n$ we have
$$V(A_1, \ldots, A_m, K_1, \ldots, K_{n-m}) \geq V(A_1^\Phi, \ldots, A_m^\Phi, K_1, \ldots, K_{n-m}).$$

(ii) For every functions $f_1, \ldots, f_m \in \mathrm{QC}_0(\mathbb{R}^n)$ with compact level sets we have
$$V(f_1, \ldots, f_m, \mathbf{1}_{K_1}, \ldots, \mathbf{1}_{K_{n-m}}) \geq V(f_1^\Phi, \ldots, f_m^\Phi, \mathbf{1}_{K_1}, \ldots, \mathbf{1}_{K_{n-m}}).$$

Proof. For (i), notice that the right hand side is actually

$$V\left(\left(\frac{\Phi(A_1)}{\Phi(D)}\right)^{\frac{1}{m}} D, \ldots, \left(\frac{\Phi(A_m)}{\Phi(D)}\right)^{\frac{1}{m}} D, K_1, \ldots, K_{n-m}\right),$$

which is equal to

$$\prod_{i=1}^m \left(\frac{\Phi(A_i)}{\Phi(D)}\right)^{\frac{1}{m}} \cdot V(D, D, \ldots, D, K_1, \ldots, K_{n-m}) = \prod_{i=1}^m \Phi(A_i)^{\frac{1}{m}}.$$

Therefore the inequality we need to prove is

$$V(A_1, \ldots A_m, K_1, \ldots, K_{n-m}) \geq \prod_{i=1}^m \Phi(A_i)^{\frac{1}{m}},$$

which is exactly the Alexandrov-Fenchel inequality. Hence the result holds (sometimes the Alexandrov-Fenchel inequality is stated only for $m = 2$, but the general case is also well known and follows by induction).

For (ii), we calculate and get

$$\begin{aligned}
V(f_1^\Phi, \ldots, f_m^\Phi, \mathbf{1}_{K_1}, \ldots, \mathbf{1}_{K_{n-m}}) &= \int_0^1 V(\overline{K}_t(f_1^\Phi), \ldots, \overline{K}_t(f_m^\Phi), K_1, \ldots, K_{n-m})dt \\
&= \int_0^1 V(\overline{K}_t(f_1)^\Phi, \ldots, \overline{K}_t(f_m)^\Phi, K_1, \ldots, K_{n-m})dt \\
&\leq \int_0^1 V(\overline{K}_t(f_1), \ldots, \overline{K}_t(f_m), K_1, \ldots, K_{n-m})dt \\
&= V(f_1, \ldots, f_m, \mathbf{1}_{K_1}, \ldots, \mathbf{1}_{K_{n-m}}).
\end{aligned}$$

This completes the proof. □

The case $m = n$ and $\Phi = \mathrm{Vol}$ in the last proposition is especially elegant, so we will state it as a corollary:

Corollary 20. *For every functions $f_1, \ldots, f_n \in \mathrm{QC}_0(\mathbb{R}^n)$ with compact level sets we have*

$$V(f_1, f_2, \ldots, f_n) \geq V(f_1^*, f_2^*, \ldots, f_n^*).$$

Notice that the last corollary generalizes the isoperimetric inequality. In fact, one may define a generalized surface area as

$$S^{(g)}(f) = V(\underbrace{f, f, \dots, f}_{n-1 \text{ times}}, g),$$

where g is some fixed rotation invariant quasi-concave function (natural candidates may be the exponential function $g(x) = e^{-|x|}$and the Gaussian $g(x) = e^{-|x|^2/2}$). From Corollary 20 it follows immediately that $S^{(g)}(f) \geq S^{(g)}(f^*)$ for every $f \in \mathrm{QC}_0(\mathbb{R}^n)$.

Remark 21. It is clear that one may work with even more general size functionals. A natural candidate seems to be

$$\Phi(A) = V(\underbrace{\mathbf{1}_A, \mathbf{1}_A, \dots, \mathbf{1}_A}_{m \text{ times}}, g_1, g_2, \dots, g_{n-m}),$$

for some fixed quasi-concave functions $g_1, \dots, g_{n-m} \in \mathrm{QC}_0(\mathbb{R}^n)$. There are, however, some major difficulties. The main problem is that if we extend such a Φ to $\mathrm{QC}_0(\mathbb{R}^n)$ in the standard way, we do not necessarily have

$$\Phi(f) = \Phi\left(f^\Phi\right)$$

Thus it is difficult to think of f^Φ as a rearrangement of f in any real sense.

However, assume that the functions g_i satisfy $0 < \int g_i < \infty$ and have homothetic level sets, i.e. $\overline{K}_t(g_i) = c_i(t) \cdot K_i$ for some function $c_i(t)$ and some $K_i \in \mathcal{K}^n$. If we define

$$\Psi(A) = V(A, A, \dots, A, K_1, K_2, \dots, K_{n-m}),$$

then for every $A \in \mathcal{K}^n$ we get

$$\begin{aligned}\Phi(A) &= \int_0^1 V(A, \dots, A, \overline{K}_t(g_1), \dots, \overline{K}_t(g_{n-m}))dt \\ &= \int_0^1 V(A, \dots, A, K_1, \dots, K_{n-m}) \cdot c_1(t)c_2(t)\cdots c_{n-m}(t)dt \\ &= \left[\int_0^1 c_1(t)c_2(t)\cdots c_{n-m}(t)dt\right] \cdot \Psi(A) = C \cdot \Psi(A).\end{aligned}$$

Since $0 < \int g_i < \infty$ we have $0 < C < \infty$, and it follows immediately that $A^\Phi = A^\Psi$. Hence $f^\Phi = f^\Psi$ for all $f \in \mathrm{QC}_0(\mathbb{R}^n)$.

Since

$$\begin{aligned}\Phi(f) &= V(f, \dots, f, g_1, \dots, g_{n-m}) \\ &= \int_0^1 V(\overline{K}_t(f), \dots, \overline{K}_t(f), \overline{K}_t(g_1), \dots, \overline{K}_t(g_{n-m}))dt \\ &= \int_0^1 c_1(t)\cdots c_{n-m}(t) \cdot V(\overline{K}_t(f), \dots, \overline{K}_t(f), K_1, \dots, K_{n-m})dt \\ &= \int_0^1 c_1(t)\cdots c_{n-m}(t) \cdot \Psi\left(\overline{K}_t(f)\right) dt,\end{aligned}$$

and similarly

$$\begin{aligned}\Phi(f^\Phi) = \Phi(f^\Psi) &= \int_0^1 c_1(t)\cdots c_{n-m}(t)\cdot \Psi\left(\overline{K}_t(f^\Psi)\right) dt \\ &= \int_0^1 c_1(t)\cdots c_{n-m}(t)\cdot \Psi\left(\overline{K}_t(f)^\Psi\right) dt,\end{aligned}$$

we conclude that in this specific case we do have $\Phi(f^\Phi) = \Phi(f)$.

Similarly, Propositions 17 and 19 remain true in this extended case:

Proposition 22. *Let*

$$\Phi(A) = V(\underbrace{\mathbf{1}_A, \mathbf{1}_A, \ldots, \mathbf{1}_A}_{m \text{ times}}, g_1, g_2, \ldots, g_{n-m}),$$

be a generalized size functional, with $g_i \in \mathrm{QC}_0(\mathbb{R}^n)$ having homothetic level sets. Then for every geometric quasi-concave functions $f, g \in \mathrm{QC}_0(\mathbb{R}^n)$ with compact level sets we have

$$(f \oplus g)^\Phi \ge f^\Phi \oplus g^\Phi.$$

Proposition 23. *Let Φ be a generalized size functional like in Proposition 22. Then for every functions $f_1, \ldots, f_m \in \mathrm{QC}_0(\mathbb{R}^n)$ with compact level sets we have*

$$V(f_1, \ldots, f_m, g_1, \ldots, g_{n-m}) \ge V(f_1^\Phi, \ldots, f_m^\Phi, g_1, \ldots, g_{n-m}).$$

The proofs are simple, as one may simply replace Φ with Ψ. We leave the details to the reader.

4. Inequalities for log-concave functions

We now turn our attention to the log-concave case. It turns out that for functions which are both geometric and log-concave, one can use some 1-dimensional estimates, and prove some of the inequalities of the previous section in a more familiar form.

First, we will need to know that the class of log-concave functions is preserved under rearrangements.

Proposition 24. *Let Φ be a size functional. If f is log-concave, so is f^Φ.*

Proof. One can express log-concavity in terms of level-sets. A function f is log-concave if and only if

$$\lambda\overline{K}_t(f) + (1-\lambda)\overline{K}_s(f) \subseteq \overline{K}_{t^\lambda s^{1-\lambda}}(f)$$

for every $s, t > 0$ and every $0 < \lambda < 1$.

Using Proposition 17(i), we get

$$\begin{aligned}\lambda\overline{K}_t(f^\Phi) + (1-\lambda)\overline{K}_s(f^\Phi) &= \lambda\overline{K}_t(f)^\Phi + (1-\lambda)\overline{K}_s(f)^\Phi \\ &\subseteq \left[\lambda\overline{K}_t(f) + (1-\lambda)\overline{K}_s(f)\right]^\Phi \\ &\subseteq \overline{K}_{t^\lambda s^{1-\lambda}}(f)^\Phi = \overline{K}_{t^\lambda s^{1-\lambda}}(f^\Phi),\end{aligned}$$

so f^Φ is indeed log-concave. □

Next, we will need a 1-dimensional moment estimate for log-concave functions:

Proposition 25. *Let $f : [0,\infty) \to [0,1]$ be a log-concave function with $f(0) = 1$. Then for every $0 < k < m$ we have*

$$\left[\frac{1}{\Gamma(m+1)}\int_0^\infty x^m f(x)dx\right]^{\frac{1}{m+1}} \le \left[\frac{1}{\Gamma(k+1)}\int_0^\infty x^k f(x)dx\right]^{\frac{1}{k+1}},$$

with equality if and only if $f(x) = e^{-cx}$ for some $c > 0$.

Proof. A known result ([3], see also [2]) states that if $f : [0,\infty) \to [0,\infty)$ is log-concave, then the function

$$\varphi(p) = \frac{1}{\Gamma(p+1)}\int_0^\infty x^p f(x)dx$$

is log-concave on $(-1,\infty)$. Since $\varphi(p) \to f(0) = 1$ as $p \to -1$, we get

$$\varphi(k) = \varphi\left(\frac{k+1}{m+1}\cdot m + \frac{m-k}{m+1}\cdot(-1)\right) \ge \varphi(m)^{\frac{k+1}{m+1}} \cdot 1^{\frac{m-k}{m+1}}.$$

Hence

$$\varphi(k)^{\frac{1}{k+1}} \ge \varphi(m)^{\frac{1}{m+1}},$$

which is what we wanted. □

We are ready to prove the Alexandrov inequalities for geometric, log-concave functions:

Theorem 26. *Define $g(x) = e^{-|x|}$. For every $f \in \mathrm{LC}_0(\mathbb{R}^n)$ and every integers $0 \le k < m < n$, we have*

$$\left(\frac{W_k(f)}{W_k(g)}\right)^{\frac{1}{n-k}} \le \left(\frac{W_m(f)}{W_m(g)}\right)^{\frac{1}{n-m}},$$

with equality if and only if $f(x) = e^{-c|x|}$ for some $c > 0$.

Proof. By Proposition 18, $W_m(f^{W_k}) \le W_m(f^{W_m}) = W_m(f)$, while $W_k(f^{W_k}) = W_k(f)$. By Proposition 24, $f^{W_k} \in \mathrm{LC}_0(\mathbb{R}^n)$ as well. Therefore we may replace f by f^{W_k} and assume without loss of generality that f is rotation invariant.

Write $f(x) = h(|x|)$, where $h : [0,\infty) \to [0,1]$ is a geometric, log-concave function. Let us express $W_k(f)$ and $W_m(f)$ in terms of h. For every $\varepsilon > 0$ we have

$$f_\varepsilon(x) = \begin{cases} 1 & |x| \le \varepsilon \\ f\left(x - \varepsilon\frac{x}{|x|}\right) & |x| > \varepsilon \end{cases} = \begin{cases} 1 & |x| \le \varepsilon \\ h(|x| - \varepsilon) & |x| > \varepsilon. \end{cases}$$

Integrating using polar coordinates, we get

$$\begin{aligned} \int f_\varepsilon &= n\omega_n\left[\int_0^\varepsilon 1\cdot r^{n-1}dr + \int_\varepsilon^\infty h(r-\varepsilon)r^{n-1}dr\right] \\ &= n\omega_n\left[\frac{\varepsilon^n}{n} + \int_0^\infty h(r)(r+\varepsilon)^{n-1}dr\right] \\ &= \omega_n\varepsilon^n + n\omega_n\cdot\sum_{i=0}^{n-1}\int_0^\infty h(r)\cdot\binom{n-1}{i}r^{n-i-1}dr\cdot\varepsilon^i, \end{aligned}$$

where $\omega_n = \mathrm{Vol}\,(D)$ is the volume of the unit ball. Comparing this with the definition of the W_i's as

$$\int f_\varepsilon = \sum_{i=0}^{n} \binom{n}{i} W_i(f)\varepsilon^i,$$

we see that for every $0 \le i < n$ we have

$$W_i(f) = \frac{n\omega_n \binom{n-1}{i} \cdot \int_0^\infty h(r) \cdot r^{n-i-1} dr}{\binom{n}{i}} = (n-i)\,\omega_n \int_0^\infty h(r) \cdot r^{n-i-1} dr.$$

Now we use Proposition 25 with k and m replaced with $n-m-1$ and $n-k-1$. We get

$$\left[\frac{1}{\Gamma(n-k)} \int_0^\infty r^{n-k-1} h(r) dr\right]^{\frac{1}{n-k}} \le \left[\frac{1}{\Gamma(n-m)} \int_0^\infty r^{n-m-1} h(r) dr\right]^{\frac{1}{n-m}}$$

or

$$\left[\frac{W_k(f)}{(n-k)\,\omega_n \Gamma(n-k)}\right]^{\frac{1}{n-k}} \le \left[\frac{W_m(f)}{(n-m)\omega_n \Gamma(n-m)}\right]^{\frac{1}{n-m}}.$$

For the function $g(x) = e^{-|x|}$ we know we have an equality in Proposition 25 , so

$$\left[\frac{W_k(g)}{(n-k)\,\omega_n \Gamma(n-k)}\right]^{\frac{1}{n-k}} = \left[\frac{W_m(g)}{(n-m)\omega_n \Gamma(n-m)}\right]^{\frac{1}{n-m}}.$$

Dividing the equations, we get

$$\left(\frac{W_k(f)}{W_k(g)}\right)^{\frac{1}{n-k}} \le \left(\frac{W_m(f)}{W_m(g)}\right)^{\frac{1}{n-m}}.$$

Finally, by the equality cases of Propositions 18 and 25, we get an equality if and only if f is rotation invariant and $h(r) = e^{-cr}$, which means that $f(x) = e^{-c|x|}$. $\square$

In the case $k = 0$, $m = 1$, we immediately obtain a sharp isoperimetric inequality (remember that, by definition, $S(f) = n \cdot W_1(f)$):

Proposition 27. *For every $f \in \mathrm{LC}_0(\mathbb{R}^n)$ we have*

$$S(f) \ge \left(\int f\right)^{\frac{n-1}{n}} \cdot \frac{S(g)}{\left(\int g\right)^{\frac{n-1}{n}}}$$

with equality if and only if $f(x) = e^{-c|x|}$ for some $c > 0$.

Remark 28. In Proposition 27 we made two assumptions about f: it must be log-concave, and it must be geometric. Both assumptions are absolutely crucial, as we will now see.

Define $f : \mathbb{R}^2 \to [0,\infty)$ by $f(x) = a^2 e^{-a|x|}$. The function f is log-concave, but not geometric unless $a = 1$. Strictly speaking, we only defined the quermassintegrals for geometric functions, but from the proof of Theorem 26 we immediately see that $\int f_\varepsilon$ is a polynomial in ε, and the coefficients are

$$\int f = W_0(f) = 2\pi \cdot \int_0^\infty a^2 e^{-ar} \cdot r dr = 2\pi,$$

while

$$S(f) = 2W_1(f) = 2\pi \cdot \int_0^\infty a^2 e^{-ar} = 2\pi a.$$

By taking $a \to 0$ we see that it is indeed impossible to get any lower bound on $S(f)$ in terms of $\int f$.

Similarly, for $a > 2$ define $f : \mathbb{R}^2 \to [0, \infty)$ by $f(x) = \left(1 + \frac{|x|}{\sqrt{a^2-3a+2}}\right)^{-a}$. The function f is geometric and quasi-concave, but not log-concave. Again using the same formulas we get

$$\begin{aligned} \int f &= 2\pi \cdot \int_0^\infty r \left(1 + \frac{r}{\sqrt{a^2 - 3a + 2}}\right)^{-a} dr = 2\pi \\ S(f) &= 2\pi \cdot \int_0^\infty \left(1 + \frac{r}{\sqrt{a^2 - 3a + 2}}\right)^{-a} dr = 2\pi \cdot \sqrt{\frac{a-2}{a-1}}. \end{aligned}$$

Taking $a \to 2^+$, we see that it is again impossible to bound $S(f)$ from below using $\int f$.

Remark 29. We stated Theorem 26 and Proposition 27 for log-concave functions, but similar results can also be stated for α-concave functions, for every non-positive value of α (see [4] for definitions) . In the class of α-concave functions, the extremal function will not be $g(x) = e^{-|x|}$, but $g(x) = (1 - \alpha\,|x|)^{1/\alpha}$. Since we have not discussed α-concave functions in this paper, and since the generalized proofs are almost identical to the ones we gave, we will not pursue this point any further.

5. Rescalings and dilations

In this last section, we will explore the notion of rescaling, discussed in the introduction. We formally define:

Definition 30. A rescaling of a function $f \in \mathrm{QC}_0(\mathbb{R}^n)$ is a function of the form $\alpha \circ f$, where $\alpha : [0, 1] \to [0, 1]$ is an increasing bijection.

It is easy to see that if $\tilde{f} = \alpha \circ f$ is a rescaling of f, then

$$\overline{K}_t(\tilde{f}) = \overline{K}_{\alpha^{-1}(t)}(f).$$

Rescaling will be especially effective if the function f satisfies certain regularity assumptions. For concreteness, let us define:

Definition 31. A function $f \in \mathrm{QC}_0(\mathbb{R}^n)$ is called regular if

(i) f is continuous.
(ii) $f(\lambda x) > f(x)$ for all $x \in \mathbb{R}^n$ and $0 < \lambda < 1$.
(iii) $f(x) \to 0$ as $|x| \to \infty$.

We will need the following technical lemma:

Lemma 32. *Let Φ be a size functional, and let $f \in \mathrm{QC}_0(\mathbb{R}^n)$ be regular. Then the map $\varphi_f : [0, 1] \to [0, \infty]$ defined by*

$$\varphi_f(t) = \Phi\left(\overline{K}_t(f)\right)$$

is a decreasing bijection.

Proof. First we notice that for every $0 < t \leq 1$, the set $\overline{K}_t(f)$ is compact: it is closed because f is continuous, and bounded because $f(x) \to 0$ as $|x| \to \infty$. Also, for every $0 \leq t < 1$, the set $\overline{K}_t(f)$ has non-empty interior, because it contains an ε-neighborhood of 0.

Now let us show that φ_f is strictly decreasing. Fix $0 < s < t < 1$. Then $\overline{K}_t(f)$ is compact, $\{x :\ f(x) \leq s\}$ is closed and these two sets are disjoint. It follows that they are ε-separated for some $\varepsilon > 0$, i.e.

$$\left[\overline{K}_t(f) + \varepsilon B_2^n\right] \cap \{x :\ f(x) \leq s\} = \emptyset.$$

This implies that

$$\overline{K}_t(f) + \varepsilon B_2^n \subseteq \{x :\ f(x) > s\} \subseteq \overline{K}_s(f),$$

so $\varphi_f(t) = \Phi\left(\overline{K}_t(f)\right) < \Phi\left(\overline{K}_s(f)\right) = \varphi_f(s)$ like we wanted.

We still need to check the end points of $[0, 1]$. For $t = 0$ we get

$$\varphi_f(0) = \Phi\left(\overline{K}_0(f)\right) = \Phi\left(\mathbb{R}^n\right) = \infty,$$

but if $t > 0$ then $\varphi_f(t) < \infty$ because $\overline{K}_t(f)$ is compact. Similarly, from the definition of regularity we see that $f(x) < f(0) = 1$ for all $x \neq 0$, so $\overline{K}_1(f) = \{0\}$. Hence we get

$$\varphi_f(1) = \Phi\left(\overline{K}_1(f)\right) = \Phi\left(\{0\}\right) = 0,$$

but if $t < 1$ then $\varphi_f(t) > 0$ since $\overline{K}_t(f)$ has non-empty interior. This completes the proof that φ_f is strictly decreasing, hence injective.

Now we wish to prove that φ_f is continuous. To do so we will need the observation that for every $0 \leq t \leq 1$

$$\{x :\ f(x) > t\} = \operatorname{int}\left[\overline{K}_t(f)\right],$$

where int denotes the topological interior. Indeed, the inclusion $\subseteq$ is obvious since the set $\{x :\ f(x) > t\}$ is open. For the other inclusion, assume $x \in \operatorname{int}\left[\overline{K}_t(f)\right]$, then $(1 + \varepsilon)x \in \overline{K}_t(f)$ for small enough $\varepsilon > 0$. This implies that

$$f(x) = f\left(\frac{1}{1+\varepsilon} \cdot (1+\varepsilon)\, x\right) > f\left((1+\varepsilon)x\right) \geq t,$$

so we proved the claim.

Now continuity follows easily: from the left we have

$$\bigcap_{s<t} \overline{K}_s(f) = \overline{K}_t(f),$$

so by continuity of classic mixed volumes we get

$$\lim_{s\to t^-} \varphi_f(t) = \lim_{s\to t^-} \Phi\left(\overline{K}_s(f)\right) = \Phi\left(\overline{K}_t(f)\right) = \varphi_f(t).$$

Similarly, from the right, we get

$$\bigcup_{s>t} \overline{K}_s(f) = \{x :\ f(x) > t\} = \operatorname{int}\left[\overline{K}_t(f)\right],$$

and again by continuity of mixed volumes we get $\lim_{s\to t^+} \varphi_f(s) = \varphi_f(t)$.

Since φ_f is continuous the image $\varphi_f\left([0, 1]\right)$ is connected, and since we already saw that $0, \infty \in \varphi_f\left([0, 1]\right)$ it follows that φ_f is onto. Hence our proof is complete. $\square$

Using the lemma, we can achieve the goals promised in the introduction. Specifically, we prove the following generalized Brunn-Minkowski inequality:

Proposition 33. *Assume $f, g \in \mathrm{QC}_0(\mathbb{R}^n)$ are regular, and fix a size functional Φ of degree k. Then one can rescale f to a function $\tilde{f}$ in such a way that*

$$\Phi(\widetilde{f} \oplus g)^{\frac{1}{k}} \geq \Phi(\widetilde{f})^{\frac{1}{k}} + \Phi(g)^{\frac{1}{k}}.$$

Proof. Using the notation of Lemma 32, define $\alpha : [0,1] \to [0,1]$ by $\alpha = \varphi_g^{-1} \circ \varphi_f$. By the lemma α is an increasing bijection, so $\widetilde{f} = \alpha \circ f$ is a rescaling of f. By direct calculation

$$\begin{aligned} \Phi\left(\overline{K}_t(\widetilde{f})\right) &= \Phi\left(\overline{K}_{\alpha^{-1}(t)}(f)\right) = \varphi_f\left(\alpha^{-1}(t)\right) = \left[\varphi_f \circ \varphi_f^{-1} \circ \varphi_g\right](t) \\ &= \varphi_g(t) = \Phi\left(\overline{K}_t(g)\right), \end{aligned}$$

so $\widetilde{f}^{\Phi} = g^{\Phi}$. By Proposition 17 we get

$$\begin{aligned} \Phi(\widetilde{f} \oplus g)^{\frac{1}{k}} &\geq \Phi(\widetilde{f}^{\Phi} \oplus g^{\Phi})^{\frac{1}{k}} = \Phi(2 \odot g^{\Phi})^{\frac{1}{k}} = 2\Phi(g^{\Phi})^{\frac{1}{k}} \\ &= \Phi(g^{\Phi})^{\frac{1}{k}} + \Phi(g^{\Phi})^{\frac{1}{k}} = \Phi(\widetilde{f}^{\Phi})^{\frac{1}{k}} + \Phi(g^{\Phi})^{\frac{1}{k}} \\ &= \Phi(\widetilde{f})^{\frac{1}{k}} + \Phi(g)^{\frac{1}{k}}. \end{aligned}$$

□

Notice that we only needed to rescale one of the functions (in this case f), but the exact rescaling depended on g. The same result can be obtained by rescaling both f and g, but in a universal way – the rescaling of f will depend on Φ but not on g, and vice versa. This is not hard to see – just choose a fixed, "universal", regular quasi-concave function h, and use the same technique we used in the proof to rescale both f and g in such a way that $\Phi\left(\overline{K}_t(\widetilde{f})\right) = \Phi\left(\overline{K}_t(\widetilde{g})\right) = \Phi\left(\overline{K}_t(h)\right)$.

As a second remark, note that we have an extra degree of freedom which we have not used. We chose our rescaling α in such a way that $\widetilde{f}^{\Phi} = g^{\Phi}$, but for any $c > 0$ we could have chosen α to satisfy that $\widetilde{f}^{\Phi} = c \odot g^{\Phi}$, and the proof would have worked in exactly the same way. Using this degree of freedom we may for example choose $\widetilde{f}$ to satisfy $\Phi(\widetilde{f}) = \Phi(f)$, or alternatively $\int \widetilde{f} = \int f$.

In a similar way, one can obtain a version of the Alexandrov-Fenchel inequality

Proposition 34. *Assume $f_1, f_2, \ldots, f_m \in \mathrm{QC}_0(\mathbb{R}^n)$ are regular functions, and $A_1, A_2, \ldots, A_{n-m} \in \mathcal{K}^n$ are compact bodies with non-empty interior. Then one can rescale each f_i to a function $\tilde{f}_i$ such that*

$$V(\widetilde{f_1}, \widetilde{f_2}, \ldots, \widetilde{f_m}, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_{n-m}})^m \geq \prod_{i=1}^{m} V(\widetilde{f_i}, \widetilde{f_i}, \ldots, \widetilde{f_i}, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_{n-m}}).$$

Proof. We will use Lemma 32 again, this time with

$$\Phi(K) = V(K, K, \ldots, K, A_1, A_2, \ldots, A_{n-m}).$$

Fix some regular quasi-concave function h, and scale each f_i using $\alpha = \varphi_h^{-1} \circ \varphi_f$. Like before we will have $\widetilde{f_i}^{\Phi} = h^{\Phi}$ for all i. Thus, using Proposition 19 we get

$$\begin{aligned}
V(\widetilde{f_1}, \widetilde{f_2}, \ldots, \widetilde{f_m}, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_{n-m}})^m &\geq V(\widetilde{f_1}^{\Phi}, \widetilde{f_2}^{\Phi}, \ldots, \widetilde{f_m}^{\Phi}, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_{n-m}})^m \\
&= V(\widetilde{h}^{\Phi}, \widetilde{h}^{\Phi}, \ldots, \widetilde{h}^{\Phi}, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_{n-m}})^m \\
&= \prod_{i=1}^{m} V(\widetilde{h}^{\Phi}, \widetilde{h}^{\Phi}, \ldots, \widetilde{h}^{\Phi}, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_{n-m}}) \\
&= \prod_{i=1}^{m} V(\widetilde{f_i}^{\Phi}, \widetilde{f_i}^{\Phi}, \ldots, \widetilde{f_i}^{\Phi}, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_{n-m}}) \\
&= \prod_{i=1}^{m} V(\widetilde{f_i}, \widetilde{f_i}, \ldots, \widetilde{f_i}, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_{n-m}}).
\end{aligned}$$

As a corollary, we immediately get □

Corollary 35. *Assume $f_1, f_2, \ldots, f_n \in \mathrm{QC}(\mathbb{R}^n)$ are regular. Then it is possible to rescale each f_i to a function $\tilde{f}_i$ in such a way that*

$$V(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) \geq \left(\prod_{i=1}^{n} \int \tilde{f}_i \right)^{\frac{1}{n}}.$$

The idea of rescaling is simple and powerful, but unfortunately it does not apply to all functions. For example, if K is a convex body then the indicator $\mathbf{1}_K$ is definitely not regular, so we cannot use the above propositions. To conclude this paper we will now describe another procedure, similar to rescaling, which does not assume regularity. The idea is to take the function f, and dilate each level set $\overline{K}_t(f)$ to the required "size". In other words, given some size functional Φ, we want to construct a function $\tilde{f}$ such that

$$\overline{K}_t(\tilde{f}) = A(t) \cdot \overline{K}_t(f),$$

and $\Phi\left(\overline{K}_t(\tilde{f})\right) = \varphi(t)$ for some prescribed φ.

The problem is that for general quasi-concave functions f and general laws φ, such a $\tilde{f}$ may not exist. The following proposition gives one case where the existence of $\tilde{f}$ is guaranteed:

Proposition 36. *Fix a size functional $\Phi : \mathcal{K}^n \to [0, \infty]$ and a geometric log-concave function $f \in \mathrm{LC}_0(\mathbb{R}^n)$. Define $M(x) = e^{-|x|}$. Then it is possible to construct a function $\tilde{f}$ such that $\overline{K}_t(\tilde{f})$ is always homothetic to $\overline{K}_t(f)$, and*

$$\varphi_{\widetilde{f}}(t) := \Phi\left(\overline{K}_t(\widetilde{f})\right) = \Phi\left(\overline{K}_t(M)\right) = \varphi_M(t)$$

for all t. The function $\tilde{f}$ will be called a dilation of f.

Proof. Assume Φ is of degree m. The idea is to construct $\widetilde{f}$ such that

$$\overline{K}_t(\widetilde{f}) = \left(\frac{\Phi\left(\overline{K}_t(M)\right)}{\Phi\left(\overline{K}_t(f)\right)} \right)^{\frac{1}{m}} \cdot \overline{K}_t(f).$$

It is obvious that for such an $\widetilde{f}$ we will have $\varphi_{\widetilde{f}} = \varphi_M$. The only thing we need to prove is that such an $\widetilde{f}$ really exists, that is that the family of convex bodies $\left\{\overline{K}_t(\widetilde{f})\right\}$ is really the level sets of some function. This will follow easily once we prove that these level sets are monotone: if $t \le s$ then $\overline{K}_t(\widetilde{f}) \supseteq \overline{K}_s(\widetilde{f})$.

Fix $0 < t \le s \le 1$. By direct computation,

$$\overline{K}_t(M) = \left\{x : \ e^{-|x|} \ge t\right\} = \left\{x : \ |x| < \log\left(\frac{1}{t}\right)\right\} = \log\left(\frac{1}{t}\right) \cdot D.$$

Define

$$\lambda = \left(\frac{\Phi\left(\overline{K}_s(M)\right)}{\Phi\left(\overline{K}_t(M)\right)}\right)^{\frac{1}{m}} = \frac{\log\frac{1}{s} \cdot \Phi(D)^{\frac{1}{m}}}{\log\frac{1}{t} \cdot \Phi(D)^{\frac{1}{m}}} = \frac{\log\frac{1}{s}}{\log\frac{1}{t}}.$$

Notice that for every $x \in \overline{K}_t(f)$ we have

$$f(\lambda x) = f(\lambda x + (1-\lambda)0) \ge f(x)^\lambda \cdot 1^{1-\lambda} \ge t^\lambda = s,$$

so $\lambda x \in \overline{K}_s(f)$. It follows that $\lambda \overline{K}_t(f) \subseteq \overline{K}_s(f)$, so

$$\Phi\left(\overline{K}_s(f)\right) \ge \lambda^m \cdot \Phi\left(\overline{K}_t(f)\right) = \frac{\Phi\left(\overline{K}_s(M)\right)}{\Phi\left(\overline{K}_t(M)\right)} \cdot \Phi\left(\overline{K}_t(f)\right),$$

or

$$\frac{\Phi\left(\overline{K}_t(M)\right)}{\Phi\left(\overline{K}_t(f)\right)} \ge \frac{\Phi\left(\overline{K}_s(M)\right)}{\Phi\left(\overline{K}_s(f)\right)}.$$

Hence we definitely have

$$\overline{K}_t(\widetilde{f}) = \left(\frac{\Phi\left(\overline{K}_t(M)\right)}{\Phi\left(\overline{K}_t(f)\right)}\right)^{\frac{1}{m}} \cdot \overline{K}_t(f) \supseteq \left(\frac{\Phi\left(\overline{K}_s(M)\right)}{\Phi\left(\overline{K}_s(f)\right)}\right)^{\frac{1}{m}} \cdot \overline{K}_s(f) = \overline{K}_s(\widetilde{f}),$$

and the proof is complete. □

We see that if $f \in \mathrm{LC}_0(\mathbb{R}^n)$, then $\widetilde{f} \in \mathrm{QC}_0(\mathbb{R}^n)$. However, the function $\widetilde{f}$ may fail to be log-concave, as the next example shows.

Example 37. Define $f(x,y) = e^{-\left(|x|+y^2\right)} \in \mathrm{LC}_0(\mathbb{R}^2)$, and choose $\Phi = \mathrm{Vol}$. Notice that

$$\left|\left\{|x| + y^2 \le c\right\}\right| = \int_{y=-\sqrt{c}}^{\sqrt{c}} \int_{x=y^2-c}^{c-y^2} dxdy = \frac{8}{3}c^{\frac{3}{2}},$$

so

$$\left|\overline{K}_t(f)\right| = \left|\left\{e^{-\left(|x|+y^2\right)} \ge t\right\}\right| = \left|\left\{|x| + y^2 \le \log\frac{1}{t}\right\}\right| = \frac{8}{3}\left(\log\frac{1}{t}\right)^{\frac{3}{2}},$$

while

$$\left|\overline{K}_t(M)\right| = \left|\left\{e^{-\sqrt{x^2+y^2}} \ge t\right\}\right| = \left|\left\{\sqrt{x^2+y^2} \le \log\frac{1}{t}\right\}\right| = \pi\left(\log\frac{1}{t}\right)^2.$$

Therefore in this case we get

$$\begin{aligned}\overline{K}_t(\widetilde{f}) &= \left(\frac{\left|\overline{K}_t(M)\right|}{\left|\overline{K}_t(f)\right|}\right)^{\frac{1}{2}} \cdot \overline{K}_t(f) = \left(\frac{\pi \log^2 \frac{1}{t}}{\frac{8}{3}\log^{\frac{3}{2}} \frac{1}{t}}\right)^{\frac{1}{2}} \overline{K}_t(f) = C \cdot \log^{\frac{1}{4}} \frac{1}{t} \cdot K_t(f) = \\ &= \left\{(x,y) : \frac{|x|}{C \log^{\frac{1}{4}} \frac{1}{t}} + \frac{y^2}{C^2 \log^{\frac{1}{2}} \frac{1}{t}} \le \log \frac{1}{t}\right\},\end{aligned}$$

for some explicit constant C. In other words, $\widetilde{f}(x,y) = t$, where $t \in (0,1]$ is the unique solution to the equation

$$\frac{|x|}{C \log^{\frac{1}{4}} \frac{1}{t}} + \frac{y^2}{C^2 \log^{\frac{1}{2}} \frac{1}{t}} = \log \frac{1}{t}.$$

In general this is difficult to solve explicitly, but for $y = 0$ we get that $\widetilde{f}(x,0)$ is the solution of

$$\frac{|x|}{C \log^{\frac{1}{4}} \frac{1}{t}} = \log \frac{1}{t},$$

so

$$\widetilde{f}(x,0) = e^{-\widetilde{C}|x|^{\frac{4}{5}}}.$$

This is enough to conclude that $\widetilde{f}$ is not a log-concave function, even though f is.

Using this proposition, we can prove our main propositions again, with rescalings replaced by dilations. As the proofs are almost identical, we will only state the results:

Proposition 38. *Assume $f, g \in \mathrm{LC}_0(\mathbb{R}^n)$, and fix a size functional Φ of degree k. Then one can dilate f and g to functions $\tilde{f}$, $\tilde{g}$ in such a way that*

$$\Phi(\widetilde{f} \oplus \widetilde{g})^{\frac{1}{k}} \ge \Phi(\widetilde{f})^{\frac{1}{k}} + \Phi(\widetilde{g})^{\frac{1}{k}}.$$

Proposition 39. *Assume $f_1, f_2, \ldots, f_m \in \mathrm{LC}_0(\mathbb{R}^n)$, and $A_1, A_2, \ldots, A_{n-m} \in \mathcal{K}^n$ are compact bodies with non-empty interior. Then one can dilate each f_i to a function $\tilde{f}_i$ such that*

$$V(\widetilde{f_1}, \widetilde{f_2}, \ldots, \widetilde{f_m}, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_{n-m}})^m \ge \prod_{i=1}^m V(\widetilde{f_i}, \widetilde{f_i}, \ldots, \widetilde{f_i}, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_{n-m}}).$$

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References

[1] S. Artstein-Avidan and V. Milman. Hidden structures in the class of convex functions and a new duality transform. *Journal of the European Mathematical Society (JEMS)*, 13(4):975–1004, 2011.

[2] S. Bobkov and M. Madiman. Concentration of the information in data with log-concave distributions. *The Annals of Probability*, 39(4):1528–1543, 2011.

[3] C. Borell. Complements of Lyapunov's inequality. *Mathematische Annalen*, 205:323–331, 1973.

[4] H. J. Brascamp and E. H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Functional Analysis*, 22(4):366–389, 1976.

[5] A. Colesanti. What is the perimeter of a log-concave function? In *Workshop on Convexity and Asymptotic Geometric Analysis*, Montreal, April 2012.

[6] R. J. Gardner, D. Hug, and W. Weil. Operations between sets in geometry. *arXiv:1205.4327*, May 2012.

[7] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.

[8] V. Milman. Geometrization of probability. In M. Kapranov, Y. Manin, P. Moree, S. Kolyada, and L. Potyagailo, editors, *Geometry and Dynamics of Groups and Spaces*, volume 265, pages 647–667. Birkhäuser Basel, Basel, 2008.

[9] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[10] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[11] C. P. Simon and L. E. Blume. *Mathematics for Economists*. W. W. Norton & Company, 1 edition, April 1994.

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