On the Mean Width of Log-Concave Functions

Liran Rotem

Abstract In this work we present a new, natural, definition for the mean width of log-concave functions. We show that the new definition coincide with a previous one by B. Klartag and V. Milman, and deduce some properties of the mean width, including an Urysohn type inequality. Finally, we prove a functional version of the finite volume ratio estimate and the low-M^* estimate.

1 Introduction and Definitions

This paper is another step in the "geometrization of probability" plan, a term coined by V. Milman. The main idea is to extend notions and results about convex bodies into the realm of log-concave functions. Such extensions serve two purposes: Firstly, the new functional results can be interesting on their own right. Secondly, and perhaps more importantly, the techniques developed can be used to prove new results about convex bodies. For a survey of results in this area see [11].

A function $f : \mathbb{R}^n \to [0, \infty)$ is called *log-concave* if it is of the form $f = e^{-\varphi}$, where $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is a convex function. For us, the definition will also include the technical assumptions that f is upper semi-continuous and f is not identically 0. Whenever we discuss f and φ simultaneously, we will always assume they satisfy the relation $f = e^{-\varphi}$. Similar relation will be assumed for $\widetilde{f}$ and $\tilde{\varphi}$, f_k and φ_k, etc. The class of log-concave functions naturally extends the class of convex bodies: if $\emptyset \neq K \subseteq \mathbb{R}^n$ is a closed, convex set, then its characteristic function $\emptyset_K$ is a log-concave function.

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On the class of convex bodies there are two important operations. If K and T are convex bodies then their *Minkowski sum* is $K+T = \{k+t : \ k \in K, t \in T\}$. If in addition $\lambda > 0$, then the λ-*homothety* of K is $\lambda \cdot K = \{\lambda k : \ k \in K\}$. These operations extend to log-concave functions: If f and g are log-concave we define their *Asplund product* (or *sup-convolution*), to be

$$(f \star g)(x) = \sup_{x_1+x_2=x} f(x_1)g(x_2).$$

If in addition $\lambda > 0$ we define the λ-*homothety* of f to be

$$(\lambda \cdot f)(x) = f\left(\frac{x}{\lambda}\right)^{\lambda}.$$

It is easy to see that these operations extend the classical operations, in the sense that $1_K \star 1_T = 1_{K+T}$ and $\lambda \cdot 1_K = 1_{\lambda K}$ for every convex bodies K, T and every $\lambda > 0$. It is also useful to notice that if f is log-concave and $\alpha, \beta > 0$ then $(\alpha \cdot f) \star (\beta \cdot f) = (\alpha + \beta) \cdot f$. In particular, $f \star f = 2 \cdot f$.

The main goal of this paper is to define the notion of *mean width* for log-concave functions. For convex bodies, this notion requires we fix an Euclidean structure on $\mathbb{R}^n$. Once we fix such a structure we define the *support function* of a body K to be $h_K(x) = \sup_{y \in K} \langle x, y \rangle$. The function $h_K : \mathbb{R}^n \to (-\infty, \infty]$ is convex and 1-homogeneous. The mean width of K is defined to be

$$M^*(K) = \int_{S^{n-1}} h_K(\theta) d\sigma(\theta), \tag{1.1}$$

where σ is the normalized Haar measure on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \ |x| = 1\}$.

The correspondence between convex bodies and support functions is linear, in the sense that $h_{\lambda K+T} = \lambda h_K + h_T$ for every convex bodies K and T and every $\lambda > 0$. It immediately follows that the mean width is linear as well. It is also easy to check that M^* is translation and rotation invariant, so $M^*(uK) = M^*(K)$ for every isometry $u : \mathbb{R}^n \to \mathbb{R}^n$.

We will also need the equivalent definition of mean width as a *quermass-integrals*: Let $D \subseteq \mathbb{R}^n$ denote the euclidean ball. If $K \subseteq \mathbb{R}^n$ is any convex body then the n-dimensional volume $|K + tD|$ is a polynomial in t of degree n, known as the *Steiner polynomial*. More explicitly, one can write

$$|K + tD| = \sum_{i=0}^{n} \binom{n}{i} V_{n-i}(K) t^i,$$

and the coefficients $V_i(K)$ are known as the quermassintegrals of K. One can also give explicit definitions for the V_i's, and it follows that $V_1(K) = |D| \cdot M^*(K)$ (more information and proofs can be found for example in [9] or [14]). From this it's not hard to prove the equivalent definition

$$M^*(K) = \frac{1}{n\,|D|} \cdot \lim_{\varepsilon \to 0^+} \frac{|D + \varepsilon K| - |D|}{\varepsilon}. \tag{1.2}$$

This last definition is less geometric in nature, but it suits some purposes extremely well. For example, using the Brunn-Minkowski theorem (again, check [9] or [14]), one can easily deduce the *Urysohn inequality*:

$$M^*(K) \geq \left(\frac{|K|}{|D|}\right)^{\frac{1}{n}}$$

for every convex body K.

In [6], B. Klartag and V. Milman give a definition for the mean width of a log-concave function, based on definition (1.2). The role of the volume is played by Lebesgue integral (which makes sense because $\int \o_K dx = |K|$), and the euclidean ball D is replaced by a Gaussian $G(x) = e^{-\frac{|x|^2}{2}}$. The result is the following definition:

Definition 1.1. The mean width of a log-concave function f is

$$\widetilde{M}^*(f) = c_n \lim_{\varepsilon \to 0^+} \frac{\int G \star (\varepsilon \cdot f) - \int G}{\varepsilon}.$$

Here $c_n = \frac{2}{n(2\pi)^{\frac{n}{2}}}$ is a normalization constant, chosen to have $\widetilde{M}^*(G) = 1$.

Some properties of $\widetilde{M}^*$ are not hard to prove. For example, it is easy to see that $\widetilde{M}^*$ is rotation and translation invariant. It is also not hard to prove a functional Urysohn inequality:

Proposition. *If f is log-concave and $\int f = \int G$, then $\widetilde{M}^*(f) \geq \widetilde{M}^*(G) = 1$.*

The proof, that appears in [6], is similar to the standard proof for convex bodies. Instead of the Brunn-Minkowski theorem one uses its functional version, known as the Prkopa–Leindler inequality (see, e.g. [13]). For other applications, however, this definition is rather cumbersome to work with. For example, by looking at the definition it is not at all obvious that $\widetilde{M}^*$ is a linear functional. It is proven in [6] that indeed

$$\widetilde{M}^*\left((\lambda \cdot f) \star g\right) = \lambda \widetilde{M}^*(f) + \widetilde{M}^*(g),$$

but only for sufficiently regular log-concave functions f and g. These difficulties, and the fact that the definition has no clear geometric intuition, made V. Milman raise the questions of whether Definition 1.1 is the "right" definition for mean width of log-concave functions.

We would like to give an alternative definition for mean width, based on the original definition (1.1). To do so, we first need to explain what is the support function of a log-concave function, following a series of papers by S. Artstein-Avidan and V. Milman. To state their result, assume that $\mathcal{T}$ maps every (upper semi-continuous) log-concave function to its support function which is lower semi-continuous and convex. It is natural to assume that $\mathcal{T}$ is a bijection, so a log-concave function can be completely recovered from its support function. It is equally natural to assume that $\mathcal{T}$ is order preserving, that is $\mathcal{T}f \geq \mathcal{T}g$ if and only if $f \geq g$ - this is definitely the case for the standard support function defined on convex bodies. In [3] it is shown that such a $\mathcal{T}$ must be of the form

$$(\mathcal{T}f)(x) = C_1 \cdot [\mathcal{L}(-\log f)](Bx + v_0) + \langle x, v_1 \rangle + C_0$$

for constants $C_0, C_1 \in \mathbb{R}$, vectors $v_0, v_1 \in \mathbb{R}^n$ and a transformation $B \in GL_n$. Here $\mathcal{L}$ is the classical *Legendre transform*, defined by

$$(\mathcal{L}\varphi)(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - \varphi(y)).$$

We of course also want $\mathcal{T}$ to extend the standard support function. This significantly reduces the number of choices and we get that $(\mathcal{T}f)(x) = \frac{1}{C}[\mathcal{L}(-\log f)](Cx)$ for some $C > 0$. The exact choice of C is not very important, and we will choose the convenient $C = 1$. In other words, we define the support function h_f of a log-concave function f to be $\mathcal{L}(-\log f)$. Notice that the support function interacts well with the operations we defined on log-concave functions: it is easy to check that $h_{(\lambda \cdot f) \star g} = \lambda h_f + h_g$ for every log-concave functions f and g and every $\lambda > 0$ (in fact this property also completely characterizes the support function - see [2]).

We would like to define the mean width of a log-concave function as the integral of its support function with respect to some measure on $\mathbb{R}^n$. In (1.1) the measure being used is the Haar measure on S^{n-1}, but since h_K is always 1-homogeneous this is completely arbitrary: for *every* rotationally invariant probability measure μ on $\mathbb{R}^n$ one can find a constant $C_\mu > 0$ such that

$$M^*(K) = C_\mu \int_{\mathbb{R}^n} h_K(x) d\mu(x)$$

for every convex body $K \subseteq \mathbb{R}^n$. We choose to work with Gaussians:

Definition 1.2. The mean width of log-concave function f is

$$M^*(f) = \frac{2}{n} \int_{\mathbb{R}^n} h_f(x) d\gamma_n(x),$$

where γ_n is the standard Gaussian probability measure on $\mathbb{R}^n$ ($d\gamma_n = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx$).

The main result of section 2 is the fact that the two definitions given above are, in fact, the same:

Theorem 1.3. *$M^*(f) = \widetilde{M^*}(f)$ for every log-concave function f.*

This theorem gives strong indication that our definition for mean width is the "right" one.

In section 3 we present some basic properties of the functional mean width. The highlight of this section is a new proof of the functional Urysohn inequality, based on Definition 1.2. Since this definition involves no limit procedure, it is also possible to characterize the equality case:

Theorem 1.4. *For any log-concave* f

$$M^*(f) \geq 2 \log \left(\frac{\int f}{\int G} \right)^{\frac{1}{n}} + 1,$$

with equality if and only if $\int f = \infty$ *or* $f(x) = Ce^{-\frac{|x-a|^2}{2}}$ *for some* $C > 0$ *and* $a \in \mathbb{R}^n$.

Finally, in section 4, we prove a functional version of the classical low-M^* estimate (see, e.g. [10]). All of the necessary background information will be presented there, so for now we settle on presenting the main result:

Theorem 1.5. *For every* $\varepsilon < M$, *every large enough* $n \in \mathbb{N}$, *every* $f : \mathbb{R}^n \to [0, \infty)$ *such that* $f(0) = 1$ *and* $M^*(f) \leq 1$ *and every* $0 < \lambda < 1$ *one can find a subspace* $E \hookrightarrow \mathbb{R}^n$ *such that* $\dim E \geq \lambda n$ *with the following property: for every* $x \in E$ *such that* $e^{-\varepsilon n} \geq (f \star G)(x) \geq e^{-Mn}$ *one have*

$$f(x) \leq \left(C(\varepsilon, M)^{\frac{1}{1-\lambda}} \cdot G \right)(x).$$

In fact, one can take

$$C(\varepsilon, M) = C \max \left(\frac{1}{\varepsilon}, M \right).$$

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2 Equivalence of the Definitions

Our first goal is to prove that $M^*(f) = \widetilde{M}^*(f)$ for every log-concave function f. We'll start by proving it under some technical assumptions:

Lemma 2.1. *Let* $f : \mathbb{R}^n \to [0, \infty)$ *be a compactly supported, bounded, log-concave function, and assume that* $f(0) > 0$. *Then* $M^*(f) = \widetilde{M}^*(f)$.

Proof. We'll begin by noticing that

$$\begin{aligned}
\left[G\star(\varepsilon\cdot f)\right](x) &= \sup_y G(x-y)\cdot f\left(\frac{y}{\varepsilon}\right)^{\varepsilon} = \sup_y \exp\left(-\frac{|x-y|^2}{2}-\varepsilon\varphi\left(\frac{y}{\varepsilon}\right)\right) = \\
&= \sup_y \exp\left(-\frac{|x|^2}{2}+\langle x,y\rangle-\frac{|y|^2}{2}-\varepsilon\varphi\left(\frac{y}{\varepsilon}\right)\right) = \\
&= e^{-\frac{|x|^2}{2}}\exp\left(\sup_z\left(\langle x,\varepsilon z\rangle-\frac{|\varepsilon z|^2}{2}-\varepsilon\varphi(z)\right)\right) = \\
&= e^{-\frac{|x|^2}{2}+\varepsilon H(x,\varepsilon)},
\end{aligned}$$

where

$$H(x,\varepsilon)=\sup_z\left(\langle x,z\rangle-\varphi(z)-\varepsilon\frac{|z|^2}{2}\right)=\mathcal{L}\left(\varphi(x)+\varepsilon\frac{|x|^2}{2}\right).$$

Since the functions $\varphi(x)+\varepsilon\frac{|x|^2}{2}$ converge pointwise to φ as $\varepsilon\to 0$, it follows that $H(x,\varepsilon)\to(\mathcal{L}\varphi)(x)$ for every x in the interior of $A=\{x:\ (\mathcal{L}\varphi)(x)<\infty\}$ (see for example Lemma 3.2 (3) in [1]).

To find A, notice the following: since f is bounded there exists an $M\in\mathbb{R}$ such that $\varphi(x)>-M$ for all x. Since f is compactly supported there exists an $R>0$ such that $\varphi(x)=\infty$ if $|x|>R$. It follows that for every x

$$\begin{aligned}
(\mathcal{L}\varphi)(x) &= \sup_y\left(\langle x,y\rangle-\varphi(y)\right)=\sup_{|y|\le R}\left(\langle x,y\rangle-\varphi(y)\right) \\
&\le \sup_{|y|\le R}\left(|x|\,|y|-\varphi(y)\right)\le R\,|x|+M<\infty.
\end{aligned}$$

Therefore $A=\mathbb{R}^n$ and $H(x,\varepsilon)\to(\mathcal{L}\varphi)(x)$ for all x.

We wish to calculate

$$\begin{aligned}
\widetilde{M}^*(f) &= c_n\lim_{\varepsilon\to0^+}\frac{\int e^{-\frac{|x|^2}{2}+\varepsilon H(x,\varepsilon)}dx-\int e^{-\frac{|x|^2}{2}}dx}{\varepsilon} = \\
&= c_n\lim_{\varepsilon\to0^+}\int\frac{e^{\varepsilon H(x,\varepsilon)}-1}{\varepsilon}\cdot e^{-\frac{|x|^2}{2}}dx,
\end{aligned}$$

and to do so we would like to justify the use of the dominated convergence theorem. Notice that for every fixed t, the function $\frac{\exp(\varepsilon t)-1}{\varepsilon}$ is increasing in ε. By substituting $z=0$ we also see that for every $\varepsilon>0$

$$(\mathcal{L}\varphi)(x)\ge H(x,\varepsilon)=\sup_z\left(\langle x,z\rangle-\varphi(z)-\varepsilon\frac{|z|^2}{2}\right)\ge-\varphi(0).$$

Therefore on the one hand we get that for every $\varepsilon>0$

$$\frac{e^{\varepsilon H(x,\varepsilon)}-1}{\varepsilon} \geq \frac{e^{-\varepsilon\varphi(0)}-1}{\varepsilon} \geq \lim_{\varepsilon\to 0^+} \frac{e^{-\varepsilon\varphi(0)}-1}{\varepsilon} = -\varphi(0) > -\infty,$$

and on the other hand we get that for every $0<\varepsilon<1$

$$\frac{e^{\varepsilon H(x,\varepsilon)}-1}{\varepsilon} \leq \frac{e^{\varepsilon(\mathcal{L}\varphi)(x)}-1}{\varepsilon} \leq e^{(\mathcal{L}\varphi)(x)}-1 \leq e^{R|x|+M}.$$

Since the functions $-\varphi(0)$ and $e^{R|x|+M}$ are both integrable with respect to the Gaussian measure the conditions of the dominated convergence theorem apply, so we can write

$$\widetilde{M}^*(f) = c_n \int \lim_{\varepsilon\to 0^+} \frac{e^{\varepsilon H(x,\varepsilon)}-1}{\varepsilon} \cdot e^{-\frac{|x|^2}{2}} dx.$$

To finish the proof we calculate

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{e^{\varepsilon H(x,\varepsilon)}-1}{\varepsilon} &= \lim_{\varepsilon\to 0^+} \frac{e^{\varepsilon H(x,\varepsilon)}-1}{\varepsilon H(x,\varepsilon)} \cdot \lim_{\varepsilon\to 0^+} H(x,\varepsilon) \\ &= \lim_{\eta\to 0^+} \frac{e^{\eta}-1}{\eta} \cdot \lim_{\varepsilon\to 0^+} H(x,\varepsilon) = (\mathcal{L}\varphi)(x) = h_f(x).\end{aligned}$$

Therefore

$$\widetilde{M}^*(f) = c_n \int h_f(x) e^{-\frac{|x|^2}{2}} dx = \frac{2}{n}\int h_f(x) d\gamma_n(x) = M^*(f)$$

like we wanted. $\square$

In order to prove Theorem 1.3 in its full generality, we first need to eliminate one extreme case: usually we think of $\widetilde{M}^*(f)$ as the differentiation with respect to ε of $\int G\star(\varepsilon\cdot f)$. However, this is not always the case, since it is quite possible that $\int G\star(\varepsilon\cdot f) \not\to \int G$ as $\varepsilon\to 0^+$ (for example this happens for $f(x)=e^{-|x|}$). The next lemma characterizes this case completely:

Lemma 2.2. *The following are equivalent for a log-concave function f:*
(i) $(\mathcal{L}\varphi)(x)<\infty$ for every x.
(ii) $\int G\star[\varepsilon\cdot f]\to\int G$ as $\varepsilon\to 0^+$.

Proof. First, notice that both conditions are translation invariant: if we define $\tilde{f}=f(x-a)$ then it's easy to check that

$$(\mathcal{L}\tilde{\varphi})(x) = (\mathcal{L}\varphi)(x) + \langle x,a\rangle \tag{2.1}$$

and

$$\int \left(G\star\left[\varepsilon\cdot\tilde{f}\right]\right)(x)dx = \int (G\star[\varepsilon\cdot f])(x-a\varepsilon)dx = \int (G\star[\varepsilon\cdot f])(x)dx. \tag{2.2}$$

Therefore, since we assumed $f \not\equiv 0$, we can translate f and assume without loss of generality that $f(0) > 0$ (or $\varphi(0) < \infty$).

Assume first that condition (i) holds. In the proof of Lemma 2.1 we saw that

$$[G \star (\varepsilon \cdot f)](x) = e^{-\frac{|x|^2}{2} + \varepsilon H(x,\varepsilon)},$$

and that if $(\mathcal{L}\varphi)(x) < \infty$ for every x then $H(x,\varepsilon) \to (\mathcal{L}\varphi)(x)$ as $\varepsilon \to 0^+$. It follows that

$$\lim_{\varepsilon \to 0^+} [G \star (\varepsilon \cdot f)](x) = e^{-\frac{|x|^2}{2} + 0 \cdot (\mathcal{L}\varphi)(x)} = G(x)$$

for every x. Since the functions $G \star (\varepsilon \cdot f)$ are log-concave, we get that $\int G \star [\varepsilon \cdot f] \to \int G$ like we wanted (See Lemma 3.2 (1) in [1]).

Now assume that (i) doesn't hold. Since the set $A = \{x : (\mathcal{L}\varphi)(x) < \infty\}$ is convex, we must have $A \subseteq H$ for some half-space

$$H = \{x : \langle x, \theta \rangle \le a\}$$

(here $\theta \in S^{n-1}$ and $a > 0$). It follows that for every $t > 0$

$$\varphi(t\theta) = (\mathcal{L}\mathcal{L}\varphi)(t\theta) = \sup_{y \in H} \left[\langle y, t\theta \rangle - (\mathcal{L}\varphi)(y)\right].$$

But for every y we know that

$$(\mathcal{L}\varphi)(y) = \sup_z \left(\langle y, z \rangle - \varphi(z)\right) \ge -\varphi(0),$$

so

$$\varphi(t\theta) \le at + b$$

where $b = \varphi(0)$. Therefore

$$\begin{aligned} H(x,\varepsilon) &\ge \sup_{t>0} \left(\langle x, t\theta \rangle - \varphi(t\theta) - \varepsilon \frac{|t\theta|^2}{2}\right) \ge \sup_{t>0} \left(t\langle x, \theta \rangle - at - b - \frac{\varepsilon t^2}{2}\right) \\ &= \frac{(\langle x, \theta \rangle - a)^2}{2\varepsilon} - b, \end{aligned}$$

and then

$$\begin{aligned} \int [G \star (\varepsilon \cdot f)](x)dx &\ge e^{-b\varepsilon} \int e^{-\frac{|x|^2}{2} + \frac{(\langle x,\theta \rangle - a)^2}{2}} dx \to \int e^{-\frac{|x|^2}{2} + \frac{(\langle x,\theta \rangle - a)^2}{2}} dx \\ &> \int G. \end{aligned}$$

It follows that we can't have convergence in (ii) and we are done. $\square$

The last ingredient we need is a monotone convergence result which may be interesting on its own right:

Proposition 2.3. *Let f be a log-concave function such that $(\mathcal{L}\varphi)(x) < \infty$ for all x. Assume that (f_k) is a sequence of log-concave functions such that for every x*

$$f_1(x) \le f_2(x) \le f_3(x) \le \cdots$$

and $f_k(x) \to f(x)$. Then:
(i) $M^*(f_k) \to M^*(f)$.
(ii) $\widetilde{M}^*(f_k) \to \widetilde{M}^*(f)$.

Proof. (i) By our assumption $\varphi_k(x) \to \varphi(x)$ pointwise. Since we assumed that$(\mathcal{L}\varphi)(x) < \infty$ it follows that $\mathcal{L}\varphi_k$ converges pointwise to $\mathcal{L}\varphi$ (again, Lemma 3.2 (3) in [1]). Now one can apply the monotone convergence theorem and get that

$$M^*(f_k) = \frac{2}{n}\int (\mathcal{L}\varphi_k)(x)d\gamma_n(x) \to \frac{2}{n}\int (\mathcal{L}\varphi)(x)d\gamma_n(x) = M^*(f),$$

like we wanted.

(ii) For $\varepsilon > 0$ define

$$F_k(\varepsilon) = \int G \star [\varepsilon \cdot f_k]$$

and

$$F(\varepsilon) = \int G \star [\varepsilon \cdot f].$$

It was observed already in [6] that F_k and F are log-concave. By our assumption on f and Lemma 2.2, F_k and F will be (right) continuous at $\varepsilon = 0$ if we define $F_k(0) = F(0) = \int G$. We would first like the show that F_k converges pointwise to F. Because all of the functions involved are log-concave, it is enough to prove that for a fixed $\varepsilon > 0$ and $x \in \mathbb{R}^n$

$$(G \star [\varepsilon \cdot f_k])(x) \to (G \star [\varepsilon \cdot f])(x)$$

(Lemma 3.2 (1) in [1]). Since $f_k \le f$ for all k it is obvious that $\lim (G \star [\varepsilon \cdot f_k])(x) \le (G \star [\varepsilon \cdot f])(x)$. For the other direction, choose $\delta > 0$. There exists $y_\delta \in \mathbb{R}^n$ such that

$$\begin{aligned}(G \star [\varepsilon \cdot f])(x) &\le G(x - y_\delta) f\left(\frac{y_\delta}{\varepsilon}\right)^\varepsilon + \delta \\ &= \lim_{k\to\infty} G(x - y_\delta) f_k\left(\frac{y_\delta}{\varepsilon}\right)^\varepsilon + \delta \le \lim_{k\to\infty} (G \star [\varepsilon \cdot f_k])(x) + \delta.\end{aligned}$$

Finally taking $\delta \to 0$ we obtain the result.

We are interested in calculating $\widetilde{M}^*(f) = c_n F'(0)$ (the derivative here is right-derivative, but it won't matter anywhere in the proof). Since F is log-concave, it will be easier for us to compute $(\log F)'(0) = \frac{F'(0)}{\int G}$. Indeed, notice that

$$\begin{aligned}(\log F)'(0) &= \sup_{\varepsilon>0}\frac{(\log F)(\varepsilon)-(\log F)(0)}{\varepsilon} = \sup_{\varepsilon>0}\sup_{k}\frac{(\log F_k)(\varepsilon)-(\log F_k)(0)}{\varepsilon}\\ &= \sup_{k}\sup_{\varepsilon>0}\frac{(\log F_k)(\varepsilon)-(\log F_k)(0)}{\varepsilon} = \sup_{k}(\log F_k)'(0)\\ &= \sup_{k}\frac{F_k'(0)}{\int G}.\end{aligned}$$

Since the sequence $F_k'(0)$ is monotone increasing we get that

$$\widetilde{M}^*(f) = c_n\int G\cdot(\log F)'(0) = \lim_{k\to\infty} c_n F_k'(0) = \lim_{k\to\infty}\widetilde{M}^*(f_k)$$

like we wanted. □

Now that we have all of the ingredients, it is fairly straightforward to prove the main result of this section:

Theorem 1.3. *$M^*(f) = \widetilde{M}^*(f)$ for every log-concave function f.*

Proof. Let $f:\mathbb{R}^n\to[0,\infty)$ be a log-concave function. By equations (2.1) and (2.2) we see that both M^* and $\widetilde{M}^*$ are translation invariant. Hence we can translate f and assume without loss of generality that $f(0)>0$.

If there exists a point x_0 such that $(\mathcal{L}\varphi)(x_0)=\infty$, then $\mathcal{L}\varphi=\infty$ on an entire half-space, so $M^*(f)=\infty$. By Lemma 2.2 we know that $\int G\star[\varepsilon\cdot f]\not\to\int G$, and then $\widetilde{M}^*(f)=\infty$ as well and we get an equality.

If $(\mathcal{L}\varphi)(x)<\infty$ for all x we define a sequence of functions $\{f_k\}_{k=1}^{\infty}$ as

$$f_k = \min(f\cdot \emptyset_{|x|\le k}, k).$$

Every f_k is log-concave, compactly supported, bounded and satisfies

$$f_k(0) = \min(f(0),k) > 0.$$

Therefore we can apply Lemma 2.1 and conclude that $M^*(f_k)=\widetilde{M}^*(f_k)$. Since the sequence $\{f_k\}$ is monotone and converges pointwise to f we can apply proposition 2.3 and get that

$$M^*(f) = \lim_{k\to\infty}M^*(f_k) = \lim_{k\to\infty}\widetilde{M}^*(f_k) = \widetilde{M}^*(f),$$

so we are done. □

3 Properties of the Mean Width

We start by listing some basic properties of the mean width, all of which are almost immediate from the definition:

Proposition 3.1. *(i) $M^*(f) > -\infty$ for every log-concave function f.*
(ii) If there exists a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) \geq 1$, then $M^*(f) \geq 0$.
(iii) M^* is linear: for every log-concave functions f, g and every $\lambda > 0$

$$M^*\left((\lambda \cdot f) \star g\right) = \lambda M^*(f) + M^*(g).$$

(iv) M^* in rotation and translation invariant.
(v) If f is a log-concave function and $a > 0$ define $f_a(x) = a \cdot f(x)$. Then

$$M^*(f_a) = M^*(f) + \frac{2}{n}\log a$$

Proof. For (i), remember we explicitly assumed that $f \not\equiv 0$, so there exists a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) > 0$. Hence

$$h_f(x) = \sup_y \left(\langle x, y\rangle - \varphi(y)\right) \geq \langle x, x_0\rangle - \varphi(x_0),$$

and then

$$M^*(f) = \frac{2}{n}\int h_f(x) d\gamma_n(x) \geq \frac{2}{n}\left[\int \langle x, x_0\rangle \, d\gamma_n(x) - \varphi(x_0)\right] = -\frac{2}{n}\varphi(x_0) > -\infty$$

like we wanted. For (ii) we know that $\varphi(x_0) < 0$, and we simply repeat the argument.

(iii) follows from the easily verified fact that the support function has the same property. In other words, if f, g are log-concave and $\lambda > 0$ then

$$h_{(\lambda \cdot f)\star g}(x) = \lambda h_f(x) + h_g(x)$$

for every x. Integrating over x we get the result.

For (iv), we already saw in the proof of Theorem 1.3 that M^* is translation invariant. For rotation invariance, notice that if u is any linear operator then

$$\begin{aligned} h_{f\circ u}(x) &= \sup_y \left[\langle x, y\rangle - \varphi(u(y))\right] = \sup_z \left[\left\langle x, u^{-1}z\right\rangle - \varphi(z)\right] = \\ &= \sup_z \left[\left\langle \left(u^{-1}\right)^* x, z\right\rangle - \varphi(z)\right] = h_f\left(\left(u^{-1}\right)^* x\right). \end{aligned}$$

In particular if u is orthogonal then $h_{f\circ u}(x) = h_f(ux)$, and the result follows since γ_n is rotation invariant.

Finally for (v), notice that $\varphi_a = \varphi - \log a$. Therefore

$$h_{f_a} = \mathcal{L}\left(\varphi - \log a\right) = \mathcal{L}\varphi + \log a = h_f + \log a,$$

and the result follows. $\square$

Remark. A comment in [6] states that $M^*(f)$ is always positive. This is not the case: from (v) we see that if f is any log-concave function with $M^*(f) < \infty$ then $M^*(f_a) \to -\infty$ as $a \to 0^+$. (ii) gives one condition that guarantees that $M^*(f) \geq 0$, and another condition can be deduced from Theorem 1.4.

We now turn our focus to the proof of Theorem 1.4, the functional Urysohn inequality. The main ingredient of the proof is the functional Santal inequality, proven in [4] for the even case and in [1] for the general case. The result can be stated as follows:

Proposition. *Let $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ be any function such that $0 < \int e^{-\varphi} < \infty$. Then, there exists $x_0 \in \mathbb{R}^n$ such that for $\tilde{\varphi}(x) = \varphi(x - x_0)$ one has*

$$\int e^{-\tilde{\varphi}} \cdot \int e^{-\mathcal{L}\tilde{\varphi}} \leq (2\pi)^n$$

We will also need the following corollary of Jensen's inequality, sometimes known as Shannon's inequality:

Proposition. *For measurable functions $p, q : \mathbb{R}^n \to \mathbb{R}$, assume the following:*
(i) $p(x) > 0$ for all $x \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} p(x)dx = 1$
(ii) $q(x) \geq 0$ for all $x \in \mathbb{R}^n$
Then

$$\int p \log \frac{1}{p} \leq \int p \log \frac{1}{q} + \log \int q,$$

with equality if and only if $q(x) = \alpha \cdot p(x)$ almost everywhere.

For a proof of this result see, e.g. Theorem B.1 in [7] (the result is stated for $n = 1$, but the proof is completely general). Using these propositions we can now prove:

Theorem 1.4. For any log-concave function f

$$M^*(f) \geq 2 \log \left(\frac{\int f}{\int G} \right)^{\frac{1}{n}} + 1,$$

with equality if and only if $\int f = \infty$ or $f(x) = Ce^{-\frac{|x-a|^2}{2}}$ for some $C > 0$ and $a \in \mathbb{R}^n$.

Proof. If $\int f = 0$ there is nothing to prove. Assume first that $\int f < \infty$. We start by applying Shannon's inequality with $p = \frac{d\gamma_n}{dx} = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}$ and $q = e^{-h_f}$:

$$\begin{aligned}
M^*(f) &= \frac{2}{n}\int h_f(x)d\gamma_n(x) = \frac{2}{n}\int p\log\frac{1}{q} \geq \frac{2}{n}\left[\int p\log\frac{1}{p} - \log\int q\right] \\
&= \frac{2}{n}\left[\int\left(\frac{|x|^2}{2}+\frac{n}{2}\log(2\pi)\right)d\gamma_n(x) - \log\left(\int e^{-h_f}\right)\right] \\
&= \int x_1^2 d\gamma_n(x) + \log(2\pi) - \frac{2}{n}\log\left(\int e^{-h_f}\right) \\
&= 1+\log(2\pi) - \frac{2}{n}\log\left(\int e^{-h_f}\right).
\end{aligned}$$

Now we wish to use the functional Santal inequality. Since the inequality we need to prove is translation invariant, we can translate f and assume without loss of generality that $x_0 = 0$. Hence we get

$$\int f \cdot \int e^{-h_f} \leq (2\pi)^n.$$

Substituting back it follows that

$$\begin{aligned}
M^*(f) &\geq 1+\log(2\pi) - \frac{2}{n}\log\left(\frac{(2\pi)^n}{\int f}\right) \\
&= 1+\frac{2}{n}\log\left(\frac{\int f}{\int G}\right),
\end{aligned}$$

which is what we wanted to prove.

From the proof we also see that equality in Urysohn inequality implies equality in Shannon's inequality. Hence for equality we must have $q(x) = \alpha \cdot p(x)$ for some constant α, or $h_f = \frac{|x|^2}{2} + a$ for some constant a. This implies that

$$\varphi = \mathcal{L}(\mathcal{L}\varphi) = \mathcal{L}\left(\frac{|x|^2}{2}+a\right) = \frac{|x|^2}{2} - a,$$

so $f(x) = Ce^{-\frac{|x|^2}{2}}$ for $C = e^{-a}$. Since we allowed translations of f in the proof, the general equality case is $f(x) = Ce^{-\frac{|x-a|^2}{2}}$ for some $C > 0$ and $a \in \mathbb{R}^n$.

Finally, we need to handle the case that $\int f = \infty$. Like in Theorem 1.3, we choose a sequence of compactly supported, bounded functions f_k such that $f_k \uparrow f$. It follows that

$$M^*(f) \geq M^*(f_k) \geq 2\log\left(\frac{\int f_k}{\int G}\right)^{\frac{1}{n}} + 1 \overset{k\to\infty}{\longrightarrow} \infty,$$

so $M^*(f) = \infty$ and we are done. $\square$

4 Low-M^* Estimate

Remember the following important result, known as the low-M^* estimate:

Theorem. *There exists a function $f : (0,1) \to \mathbb{R}^+$ such that for every convex body $K \subseteq \mathbb{R}^n$ and every $\lambda \in (0,1)$ one can find a subspace $E \hookrightarrow \mathbb{R}^n$ such that $\dim E \geq \lambda n$ and*

$$K \cap E \subseteq f(\lambda) \cdot M^*(K) \cdot D_E$$

This result was first proven by V. Milman in [8] with $f(\lambda) = C^{\frac{1}{1-\lambda}}$ for some universal constant C. Many other proofs were later found, most of which give sharper bounds on $f(\lambda)$ as $\lambda \to 1^-$ (an incomplete list includes [10], [12], and [5]).

The original proof of the low-M^* estimate passes through another result, known as the finite volume ratio estimate. Remember that if K is a convex body, then the volume ratio of K is

$$V(K) = \inf \left(\frac{|K|}{|\mathcal{E}|} \right)^{\frac{1}{n}},$$

where the infimum is over all ellipsoids $\mathcal{E}$ such that $\mathcal{E} \subseteq K$. In order to state the finite volume ratio estimate it is convenient to assume without loss of generality that this maximizing ellipsoid is the euclidean ball D. The finite volume ratio estimate ([15, 16]) then reads:

Theorem. *Assume $D \subseteq K$ and $\left(\frac{|K|}{|D|}\right)^{\frac{1}{n}} \leq A$. Then for every $\lambda \in (0,1)$ one can find a subspace $E \hookrightarrow \mathbb{R}^n$ such that $\dim E \geq \lambda n$ and*

$$K \cap E \subseteq (C \cdot A)^{\frac{1}{1-\lambda}} \cdot (D \cap E)$$

for some universal constant C. In fact, a random subspace will have the desired property with probability $\geq 1 - 2^{-n}$.

We would like to state and prove functional versions of these results. For simplicity, we will only define the functional volume ratio of a log-concave function f when $f \geq G$:

Definition 4.1. Let f be a log-concave function and assume that $f(x) \geq G(x)$ for every x. We define the relative volume ratio of f with respect to G as

$$V(f) = \left(\frac{\int f}{\int G} \right)^{\frac{1}{n}} = \frac{1}{\sqrt{2\pi}} \left(\int f \right)^{\frac{1}{n}}.$$

Theorem 4.2. *For every $\varepsilon < 1 < M$, every large enough $n \in \mathbb{N}$, every log-concave $f : \mathbb{R}^n \to [0,\infty)$ such that $f \geq G$ and every $0 < \lambda < 1$ one can find a subspace $E \hookrightarrow \mathbb{R}^n$ such that $\dim E \geq \lambda n$ with the following property: for every $x \in E$ such that $e^{-\varepsilon n} \geq f(x) \geq e^{-Mn}$ one have*

$$f(x) \le \left([C(\varepsilon, M) \cdot V(f)]^{\frac{2}{1-\lambda}} \cdot G \right)(x).$$

Here $C(\varepsilon, M)$ is a constant depending only on ε and M, and in fact we can take

$$C(\varepsilon, M)^2 = C \max\left(\frac{1}{\varepsilon}, M\right).$$

Proof. For any $\beta > 0$ *define*

$$K_{f,\beta} = \left\{ x \in \mathbb{R}^n |\, f(x) \ge e^{-\beta n} \right\}.$$

We will bound the volume ratio of $K_{f,\beta}$ in terms of $V(f)$. Because $f \ge G$ we get

$$K_{f,\beta} \supseteq K_{G,\beta} = \left\{ x \in \mathbb{R}^n |\, e^{-\frac{|x|^2}{2}} \ge e^{-\beta n} \right\} = \sqrt{2\beta n} D.$$

We will prove a simple upper bound for the volume of $K_{f,\beta}$. Since f is log-concave one get that for every $\beta_1 \le \beta_2$

$$K_{f,\beta_1} \subseteq K_{f,\beta_2} \subseteq \frac{\beta_2}{\beta_1} K_{f,\beta_1}.$$

In particular, we can conclude that for every $\beta > 0$

$$K_{f,\beta} \subseteq \max(1, \beta) \cdot K_{f,1}.$$

However, a simple calculation tells us that

$$\int f \ \ge \int_{K_{f,1}} f \ \ge |K_{f,1}| \cdot e^{-n},$$

so

$$|K_{f,\beta}| \le \max(1, \beta)^n \, |K_{f,1}| \le [e \cdot \max(1, \beta)]^n \int f.$$

Putting everything together we can bound the volume ratio for $K_{f,\beta}$ with respect to the ball $\sqrt{2\beta n} D$:

$$V(K_{f,\beta}) = \left(\frac{|K_{f,\beta}|}{\left|\sqrt{2\beta n} D\right|} \right)^{\frac{1}{n}} \le \frac{e \cdot \max(1, \beta)}{\sqrt{2\beta n}} \cdot \left(\frac{\int f}{\int G} \right)^{\frac{1}{n}} \cdot \left(\frac{\int G}{|D|} \right)^{\frac{1}{n}} \le$$
$$\le C \max(\frac{1}{\sqrt{\beta}}, \sqrt{\beta}) \cdot V(f).$$

Now we pick a one dimensional net $\varepsilon = \beta_0 < \beta_1 < ... < \beta_{N-1} < \beta_N = M$ such that $\frac{\beta_{i+1}}{\beta_i} \le 2$. Using the standard finite volume ratio theorem for convex bodies we find a subspace $E \subseteq \mathbb{R}^n$ such that

$$K_{f,\beta_i} \cap E \subseteq \left[C \max(\frac{1}{\sqrt{\beta_i}}, \sqrt{\beta_i}) \cdot V(f)\right]^{\frac{1}{1-\lambda}} \sqrt{2\beta_i n} D \subseteq$$
$$\subseteq \left[C \max(\frac{1}{\sqrt{\varepsilon}}, \sqrt{M}) \cdot V(f)\right]^{\frac{1}{1-\lambda}} \sqrt{2\beta_i n} D.$$

for every $0 \le i \le N$ (This will be possible for large enough n. In fact, it's enough to take $n \ge \log\log \frac{M}{\varepsilon}$).

For every $x \in E$ such that $e^{-\varepsilon n} \ge f(x) \ge e^{-Mn}$ pick the smallest i such that $e^{-\beta_i n} \le f(x)$. Then $x \in K_{f,\beta_i} \cap E$, and therefore

$$|x| \le \left[C \max(\frac{1}{\sqrt{\varepsilon}}, \sqrt{M}) \cdot V(f)\right]^{\frac{1}{1-\lambda}} \sqrt{2\beta_i n},$$

or

$$G(x) = e^{-\frac{|x|^2}{2}} \ge \exp\left(-(\beta_i n) \cdot \left[C \max(\frac{1}{\sqrt{\varepsilon}}, \sqrt{M}) \cdot V(f)\right]^{\frac{2}{1-\lambda}}\right) \ge$$
$$\ge \exp\left(-(\beta_{i-1} n) \cdot \left[C' \max(\frac{1}{\sqrt{\varepsilon}}, \sqrt{M}) \cdot V(f)\right]^{\frac{2}{1-\lambda}}\right).$$

This is equivalent to

$$\left(\left[C' \max(\frac{1}{\sqrt{\varepsilon}}, \sqrt{M}) \cdot V(f)\right]^{\frac{2}{1-\lambda}} \cdot G\right)(x) \ge e^{-\beta_{i-1} n} > f(x)$$

which is exactly what we wanted. □

Remark. The role of ε and M in the above theorem might seem a bit artificial, as the condition $e^{-\varepsilon n} \ge f(x) \ge e^{-Mn}$ has no analog in the classical theorem. This condition is necessary however, as some simple examples show. For example, consider $f(x) = e^{-\varphi(|x|)}$ where

$$\varphi(x) = \begin{cases} 0 & x < \sqrt{n} \\ 2\sqrt{n}x - 2n & \sqrt{n} \le x \le 2\sqrt{n} \\ \frac{x^2}{2} & 2\sqrt{n} \le x. \end{cases}$$

To explain the origin of this example notice that f is the log-concave envelope of $\max(G, \o_{\sqrt{n}D})$. It is easy to check that $f \ge G$ and $V(f)$ is bounded from above by a universal constant independent of n. Since f is rotationally invariant the role of the subspace E in the theorem is redundant, and one easily checks that $f(x) \ge e^{-\varepsilon n}$ if and only if

$$f(x) \le \left(\frac{(\varepsilon+2)^2}{8\varepsilon} \cdot G\right)(x) \sim \left(\frac{1}{2\varepsilon} \cdot G\right)(x).$$

This shows that not only does $C(\varepsilon, M)$ must depend on ε, but the dependence we showed is essentially sharp as $\varepsilon \to 0$. Similar examples show that the same is true for the dependence in M.

Using Theorem 4.2 we can easily prove Theorem 1.5 :

Theorem 1.5. For every $\varepsilon < M$, every large enough $n \in \mathbb{N}$, every $f : \mathbb{R}^n \to [0, \infty)$ such that $f(0) = 1$ and $M^*(f) \le 1$ and every $0 < \lambda < 1$ one can find a subspace $E \hookrightarrow \mathbb{R}^n$ such that $\dim E \ge \lambda n$ with the following property: for every $x \in E$ such that $e^{-\varepsilon n} \ge (f \star G)(x) \ge e^{-Mn}$ one have

$$f(x) \le \left(C(\varepsilon, M)^{\frac{1}{1-\lambda}} \cdot G \right)(x).$$

In fact, one can take

$$C(\varepsilon, M) = C \max\left(\frac{1}{\varepsilon}, M\right).$$

Proof. Define $h = f \star G$. Since $f(0) = 1$ it follows that

$$(f \star G)(x) = \sup_{x_1 + x_2 = x} f(x_1) G(x_2) \ge f(0) G(x) = G(x).$$

Since M^* is linear $M^*(h) = M^*(f) + M^*(G) \le 2$, so by theorem 1.4 we get that $V(h) \le \sqrt{e}$. Applying Theorem 4.2 for h, and noticing that $f(x) \le h(x)$ for all x, we get the result. $\square$

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