Non-standard constructions in convex geometry: Geometric means of convex bodies

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Abstract

In this note we discuss new constructions of convex bodies. By thinking of the polarity map $K \mapsto K^\circ$ as the inversion $x \mapsto x^{-1}$ one may construct new bodies which were not previously considered in convex geometry. We illustrate this philosophy by describing a recent result of Molchanov, who constructed continued fractions of convex bodies.

Our main construction is the geometric mean of two convex bodies. We define it using the above ideology, and discuss its properties and its structure. We also compare our new definition with the "logarithmic mean" of Böröczky, Lutwak, Yang and Zhang, and discuss volume inequalities. Finally, we discuss possible extensions of the theory to p-additions and to the functional case, and present a list of open problems.

An appendix to this paper, written by Alexander Magazinov, presents a 2-dimensional counterexample to a natural conjecture involving the geometric mean.

1 Introduction

A *convex body* in $\mathbb{R}^n$ is a compact, convex set $K \subseteq \mathbb{R}^n$. We will always make the additional assumption that 0 is in the interior of K, and denote the class of such convex bodies in $\mathbb{R}^n$ by $\mathcal{K}^n_{(0)}$. We also denote the (Lebesgue) volume of K by $|K|$, and the unit ball of ℓ_p^n by B_p^n.

The goal of this paper is to discuss constructions of new convex bodies out of old ones. The most well-known such construction is the *Minkowski addition*. For $K, T \in \mathcal{K}^n_{(0)}$ we define

$$K + T = \{x + y : \ x \in K, \ y \in T\}.$$

If $\lambda > 0$ and $K \in \mathcal{K}^n_{(0)}$ then the *homothety* λK is defined in the obvious way as $\lambda K = \{\lambda x : \ x \in K\}$. Once the addition and the homothety are defined, we may of course define the arithmetic mean of K and T as $A(K,T) = \frac{1}{2}(K + T)$.

Another standard construction in convex geometry is the polarity transform. The *polar* (or dual) of a body $K \in \mathcal{K}^n_{(0)}$ is defined by

$$K^\circ = \{y \in \mathbb{R}^n : \ \langle x, y\rangle \le 1 \text{ for all } x \in K\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^n$.

For an equivalent description of the polar body, remember that to every $K \in \mathcal{K}^n_{(0)}$ one may associate two standard functions $h_K, r_K : \mathbb{R}^n \to (0, \infty)$. The *support function* h_K is defined by

$$h_K(\theta) = \max_{x \in K} \langle x, \theta \rangle,$$

and the *radial function* r_K is defined by

$$r_K(\theta) = \max\{\lambda > 0 : \ \lambda\theta \in K\}.$$

As h_K is 1-homogeneous and r_K is (-1)-homogeneous, it is usually enough to think of them as functions on S^{n-1}, the unit Euclidean sphere in $\mathbb{R}^n$. We will often write $h_\theta(K)$ and $r_\theta(K)$ instead of $h_K(\theta)$ and $r_K(\theta)$, especially in situations where $\theta \in S^{n-1}$ is fixed and K changes. Each of the functions h_K and r_K determines the body K uniquely, and the polar body K° can be defined by the relation $r_\theta(K^\circ) = h_\theta(K)^{-1}$. Remember also that for every $K, T \in \mathcal{K}^n_{(0)}$ and every $\lambda > 0$ we have $h_\theta(\lambda K + T) = \lambda h_\theta(K) + h_\theta(T)$.

The polarity map $\circ : \mathcal{K}^n_{(0)} \to \mathcal{K}^n_{(0)}$ is an *abstract duality* in the sense of [3] (see also [22]). This means that it satisfies the following two properties:

- It is an *involution*: $(K^\circ)^\circ = K$ for all $K \in \mathcal{K}^n_{(0)}$.
- It is *order reversing*: If $K \supseteq T$, then $K^\circ \subseteq T^\circ$.

In fact, the polarity map is essentially the only duality on $\mathcal{K}^n_{(0)}$:

Theorem 1. *Let $\mathcal{T} : \mathcal{K}^n_{(0)} \to \mathcal{K}^n_{(0)}$ be an order reversing involution. Then there exists a symmetric and invertible linear map $u : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{T}K = u(K^\circ)$.*

This theorem essentially appears in the work of Böröczky and Schneider ([6]). A similar theorem on a different class of convex sets was proved by Artstein-Avidan and Milman ([2]). On yet another class of convex sets, the theorem can also be deduced from the work of Gruber ([13]).

There is another famous duality in mathematics: the inverse map. the map $x \mapsto x^{-1}$ defined on $\mathbb{R}_+$ is a duality in the above sense. The same is true if one replaces $\mathbb{R}_+$ with the the class $\mathcal{M}^n_+$ of $n \times n$ positive-definite matrices. For the constructions described in this paper, it will be useful to think of K° as the inverse "K^{-1}".

Let us give one example of this point of view. Once we have an inverse map and an addition operation, we can easily construct the harmonic mean: The harmonic mean of K and T is simply

$$H(K,T) = \left(\frac{K^\circ + T^\circ}{2}\right)^\circ.$$

Naturally, we expect the harmonic mean to be smaller then the arithmetic mean. This is true, and was proved by Firey in [11]:

Theorem 2 (Firey)**.** *For every $K, T \in \mathcal{K}^n_{(0)}$ one has*

$$\frac{K+T}{2} \supseteq \left(\frac{K^\circ + T^\circ}{2}\right)^\circ.$$

Since we will rely heavily on this result, we reproduce its short proof in a more modern notation:

Proof. Fix $\theta \in S^{n-1}$. Since $r_\theta(K) \cdot \theta \in K$ and $r_\theta(T) \cdot \theta \in T$ we have

$$\frac{r_\theta(K) + r_\theta(T)}{2} \cdot \theta \in \frac{K+T}{2},$$

and hence by definition

$$r_\theta\left(\frac{K+T}{2}\right) \geq \frac{r_\theta(K) + r_\theta(T)}{2}.$$

On the other hand we have

$$r_\theta\left(\left(\frac{K^\circ+T^\circ}{2}\right)^\circ\right) = \left(h_\theta\left(\frac{K^\circ+T^\circ}{2}\right)\right)^{-1}$$
$$= \left(\frac{h_\theta\left(K^\circ\right)+h_\theta\left(T^\circ\right)}{2}\right)^{-1} = \frac{2}{\frac{1}{r_\theta(K)}+\frac{1}{r_\theta(T)}}.$$

The result now follows from the arithmetic mean-harmonic mean inequality for real numbers. □

The construction of the harmonic mean is not terribly exciting, but it emerged naturally from the same philosophy as the rest of this paper. In the next section we will follow the work of Molchanov, and describe a more interesting construction – continued fractions of convex bodies. The next several sections are devoted to the geometric mean of convex bodies, the main construction of this paper. Section 8 is devoted to a possible extension of the theory to the functional case. Finally, in Section 9 we list several open problems.

2 Continued fractions of convex bodies

For a sequence of positive real numbers $\{x_m\}_{m=1}^\infty$, the continued fraction $[x_1, x_2, x_3, \ldots]$ is simply

$$\frac{1}{x_1+\frac{1}{x_2+\frac{1}{x_3+\frac{1}{\cdots}}}}.$$

More formally, we define $[x_1] = \frac{1}{x_1}$ and

$$[x_1, x_2, \ldots, x_m] = (x_1 + [x_2, x_3, \ldots, x_m])^{-1},$$

and we set

$$[x_1, x_2, x_3, \ldots] = \lim_{m\to\infty} [x_1, x_2, \ldots, x_m].$$

It is not hard to see that this sequence indeed converges if $x_m > \epsilon$ for all m and some fixed $\epsilon > 0$. In particular, the continued fraction converges whenever the x_i's are all integers.

In [18], Molchanov generalizes the construction of continued fractions to the general setting of a partially ordered abelian semigroup equipped with an abstract duality (i.e. an order reversing involution). We will only state his results for the class $\mathcal{K}^n_{(0)}$, where the duality is of course the polarity transform.

For a sequence of convex bodies $\{K_m\}_{m=1}^\infty \subseteq \mathcal{K}^n_{(0)}$ we set $[K_1] = K_1^\circ$ and

$$[K_1, K_2, \ldots, K_m] = (K_1 + [K_2, K_3, \ldots, K_m])^\circ.$$

In order to discuss the convergence of $\lim_{m\to\infty} [K_1, K_2, \ldots, K_m]$ we need a suitable metric on $\mathcal{K}^n_{(0)}$. The obvious choice, and the one used by Molchanov, is the Hausdorff distance:

$$d(K, T) = \min\left\{r > 0 : \ K \subseteq T + rB_2^n \text{ and } T \subseteq K + rB_2^n\right\}.$$

We can now state Molchanov's theorem:

Theorem 3 (Molchanov)**.** *Let $\{K_m\}_{m=1}^\infty \subseteq \mathcal{K}^n_{(0)}$ be a family of convex bodies. Assume that one of the following three conditions hold:*

1. *$K_m \supseteq rB_2^n$ for all m and for some $r > 1$.*

2. $B_2^n \subseteq K_m \subseteq R \cdot B_2^n$ *for all* m *and for some* $R < \infty$.

3. $rB_2^n \subseteq K_m \subseteq R \cdot B_2^n$ *for all* m *for some such* $r < 1$ *and* $R \leq r/(1-r)$.

Then

$$[K_1, K_2, \ldots] = \lim_{m\to\infty} [K_1, K_2, \ldots, K_m]$$

exists in the Hausdorff sense.

As a corollary of the above theorem, one can deduce the following result:

Proposition 4 (Molchanov)**.** *For every convex body* $K \in \mathcal{K}^n_{(0)}$ *such that* $K \supseteq B_2^n$ *there exists a unique body* $Z \in \mathcal{K}^n_{(0)}$ *such that* $Z^\circ = Z + K$.

Notice that if we think of Z° as the inverse "Z^{-1}", the equation $Z^\circ = Z + K$ is a "quadratic equation" of convex bodies. Its solution can be written in a continued fraction form, $Z = [K, K, K, \ldots]$, and the convergence of this fraction follows from Theorem 3. The uniqueness part of Proposition 4 does not appear in Molchanov's paper, but follows easily from his techniques.

We will now give a self-contained proof of Proposition 4. The proof is essentially Molchanov's, but since we do not strive for generality we can present the proof in a more transparent form. We begin with a lemma, also taken from Molchanov's paper:

Lemma 5 (Molchanov)**.** *If* $K, T \supseteq rB_2^n$ *then* $d(K^\circ, T^\circ) \leq r^{-2} \cdot d(K, T)$.

Proof. Write $d = d(K, T)$. By definition of the Hausdorff distance we have

$$K \subseteq T + d \cdot B_2^n \subseteq T + \frac{d}{r}T = \frac{r+d}{r}T,$$

and since polarity is order reversing it follows that

$$K^\circ \supseteq \left(\frac{r+d}{r}T\right)^\circ = \frac{r}{r+d}T^\circ.$$

Since $K \supseteq rB_2^n$ we also have $B_2^n \supseteq rK^\circ$, so

$$K^\circ + \frac{d}{r^2}B_2^n \supseteq K^\circ + \frac{d}{r^2}rK^\circ = \left(\frac{r+d}{r}\right)K^\circ \supseteq \frac{r+d}{r} \cdot \frac{r}{r+d} \cdot T^\circ = T^\circ.$$

By exchanging the roles of K and T we also have $T^\circ + \frac{d}{r^2}B_2^n \supseteq T^\circ$, so $d(K^\circ, L^\circ) \leq \frac{d}{r^2} = r^{-2}d(K, T)$. $\square$

We may now proof Proposition 4:

Proof. Define a sequence of convex bodies by

$$\begin{aligned} Z_1 &= K^\circ \\ Z_{m+1} &= (K + Z_m)^\circ. \end{aligned}$$

Our first goal is to prove that $Z_m \supseteq \epsilon B_2^n$ for all m and some fixed $\epsilon > 0$. Indeed, K is assumed to be compact so $B_2^n \subseteq K \subseteq R \cdot B_2^n$ for some $R > 0$. If we now define two sequences of real numbers $\{a_m\}_{m=1}^\infty$, $\{b_m\}_{m=1}^\infty$ by

$$\begin{aligned} a_1 &= \frac{1}{R} & b_1 &= 1 \\ a_{m+1} &= \frac{1}{R + b_m} & b_{m+1} &= \frac{1}{1 + a_m}, \end{aligned}$$

it is trivial to prove by induction that $a_m B_2^n \subseteq Z_m \subseteq b_m B_2^n$ for all m. Since

$$\lim_{m\to\infty} a_m = [R, 1, R, 1, R, 1, \ldots] > 0,$$

it follows that $a_m > \epsilon$ for all m and some fixed $\epsilon > 0$, which proves our claim.

Using the above fact and the lemma, we deduce that for every $m > 1$ we have

$$\begin{aligned} d(Z_{m+1}, Z_m) &= d\left((K+Z_m)^\circ, (K+Z_{m-1})^\circ\right) \le \left(\frac{1}{1+\epsilon}\right)^2 d\left(K+Z_m, K+Z_{m-1}\right) \\ &= \left(\frac{1}{1+\epsilon}\right)^2 d(Z_m, Z_{m-1}). \end{aligned}$$

Hence the sequence $\{Z_m\}$ is a Cauchy sequence, so the limit $Z = \lim_{m\to\infty} Z_m$ exists. In general the limit of bodies in $\mathcal{K}_{(0)}^n$ does not have to be in $\mathcal{K}_{(0)}^n$, as it may have an empty interior. In our case, however, we have $Z_m \supseteq \epsilon B_2^n$ for all m, so $Z \supseteq \epsilon B_2^n$ and $Z \in \mathcal{K}_{(0)}^n$. Sending $m \to \infty$ in the relation $Z_{m+1} = (K+Z_m)^\circ$ and using the continuity of the polarity transform we obtain $Z = (K+Z)^\circ$, so the existence part of the proposition is proved.

For the uniqueness, assume $Z, W \in \mathcal{K}_{(0)}^n$ satisfy both $Z^\circ = Z + K$ and $W^\circ = W + K$. Fix some $\epsilon > 0$ such that $Z, W \supseteq \epsilon B_2^n$. Then

$$d(Z,W) = d\left((Z+K)^\circ, (W+K)^\circ\right) \le \left(\frac{1}{1+\epsilon}\right)^2 d\left(Z+K, W+K\right) = \left(\frac{1}{1+\epsilon}\right)^2 d(Z,W),$$

so $d(Z,W) = 0$ and $Z = W$. □

Denote the unique solution of $Z^\circ = Z + K$ by $Z(K)$. Notice that $Z(rB_2^n) = [r, r, r, \ldots] \cdot B_2^n$. In particular, for $r = 1$ we have $Z(B_2^n) = \frac{1}{\varphi} B_2^n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. However, for other choices of K (say the unit cube), the body $Z(K)$ is completely mysterious, and we know very little about its properties. It appears to be a genuinely new construction in convexity.

3 The geometric mean of convex bodies

Over the recent years, there were several attempts to define the geometric mean of two convex bodies K and T. Let us recall some of these ideas, not in chronological order:

In [5], Böröczky, Lutwak, Yang and Zhang construct the following "0-mean", or "logarithmic mean", of convex bodies:

$$L_\lambda(K,T) = \left\{x \in \mathbb{R}^n : \langle x, \theta\rangle \le h_K(\theta)^{1-\lambda} h_T(\theta)^\lambda \text{ for all } \theta \in S^{n-1}\right\}.$$

In other words, the support function h_L of $L = L_\lambda(K,T)$ is the largest convex function such that $h_L(\theta) \le h_K(\theta)^{1-\lambda} h_T(\theta)^\lambda$ for all $\theta \in S^{n-1}$.

The authors of [5] conjecture that $L_\lambda(K,T)$ satisfy a Brunn-Minkowski type inequality. To describe their conjecture, let us denote by $\mathcal{K}_s^n$ the class of origin-symmetric convex bodies, i.e. the sets $K \in \mathcal{K}_{(0)}^n$ such that $K = -K$. The *log-Brunn-Minkowski conjecture* then states that for every $K, T \in \mathcal{K}_s^n$ and every $\lambda \in [0,1]$ we have $|L_\lambda(K,T)| \ge |K|^{1-\lambda} |T|^\lambda$. It is still unknown whether this conjecture is true – it was proven in [5] in dimension $n = 2$, and in [24] by Saroglou for unconditional convex bodies in $\mathbb{R}^n$.

To explain its name, notice that the log-Brunn-Minkowski conjecture is a strengthening of the classic Brunn-Minkowski inequality. Indeed, by the arithmetic mean-geometric mean inequality we have

$$h_K(\theta)^{1-\lambda} h_T(\theta)^\lambda \le (1-\lambda) h_K(\theta) + \lambda h_T(\theta) = h_{(1-\lambda)K + \lambda T}(\theta),$$

so the log-Brunn-Minkowski inequality implies that

$$|(1-\lambda)K+\lambda T| \geq |L_\lambda(K,T)| \geq |K|^{1-\lambda}\,|T|^\lambda\,,$$

which is exactly the Brunn-Minkowski inequality in its dimension free form.

Let us mention that one can also consider the "dual" construction to L_λ, where instead of the support functions one take the geometric average of the radial functions. The body obtained is simply $L_\lambda\left(K^\circ,T^\circ\right)^\circ$, and Saroglou proved in [23] that $|L_\lambda(K^\circ,T^\circ)^\circ| \leq |K|^{1-\lambda}\,|T|^\lambda$ for every $K,T\in\mathcal{K}_s^n$. By the Blaschke-Santaló inequality and Bourgain-Milman theorem ([7]), it follows that

$$|L_\lambda(K,T)| \geq c^n\,|K|^{1-\lambda}\,|T|^\lambda \tag{3.1}$$

for some universal constant $c>0$.

For complex bodies, the situation is much clearer. Notice that, by our definition, a convex body $K\in\mathcal{K}_s^n$ is simply the unit ball of a norm on $\mathbb{R}^n$. Similarly, a complex convex body $K\subseteq\mathbb{C}^n$ is the unit ball of a norm on $\mathbb{C}^n$. By identifying $\mathbb{C}^n\simeq\mathbb{R}^{2n}$ we see that every complex body is also a real body, but not vice versa. In fact, a complex body $K\subseteq\mathbb{C}^n$ is a real body which is also symmetric with respect to complex rotations, i.e. $z\in K$ implies that $e^{i\theta}z\in K$ for all $\theta\in\mathbb{R}$.

There is a standard method in the literature to interpolate between complex norms, or equivalently, between complex bodies. This method is known simply as "complex interpolation" and is described for example in chapter 7 of [19]. In [8], Cordero-Erausquin proves that for every complex bodies K and T and every $\lambda\in[0,1]$, we have the relation $|C_\lambda(K,T)| \geq |K|^{1-\lambda}\,|T|^\lambda$ where C_λ denotes the complex interpolation. From here he deduces an extension of the Blaschke-Santaló inequality: Since

$$C_{1/2}(K\cap T, K^\circ\cap T) \subseteq B_2^{2n}\cap T$$

we must have $|K\cap T|\,|K^\circ\cap T| \leq \left|B_2^{2n}\cap T\right|^2$. Cordero-Erausquin asks whether this inequality also holds for real convex bodies, and this question is still open. A partial answer was given in [14] by Klartag, who proved in the real case a functional version of the inequality. As a corollary he proved that for every $K,T\in\mathcal{K}_s^n$ we have

$$|K\cap T|\,|K^\circ\cap T| \leq 2^n\,|B_2^n\cap T|^2 \tag{3.2}$$

It is also true that for complex bodies $C_\lambda(K,T)\subseteq L_\lambda(K,T)$, so the log-Brunn-Minkowski conjecture is true for complex bodies (see [21]).

Finally, let us briefly mention a third possible "geometric mean". The construction was studied by Cordero-Erausquin and Klartag in [9], following a previous work of Semmes ([25]). Let $u_0,u_1:\mathbb{R}^n\to\mathbb{R}$ be (sufficiently smooth) convex functions. A p-interpolation between u_0 and u_1 is a function $u:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ such that $u(0,x)=u_0(x)$, $u(1,x)=u_1(x)$, and $u(t,x)$ satisfies the PDE

$$\partial_{tt}^2 u = \frac{1}{p}\left\langle \left(\mathrm{Hess}_x u\right)^{-1}\nabla\partial_t u, \nabla\partial_t u\right\rangle.$$

Here we will care about the case $p=2$. Given u_0 and u_1 it is not clear that this PDE has a solution, let alone a unique solution. However, it is not hard to check that if $u_0=\frac{1}{2}h_K^2$ and $u_1=\frac{1}{2}h_L^2$ for some bodies K and L, then $u_t=\frac{1}{2}h_{R_t}^2$ (assuming it exists) for some family of convex bodies $R_t=R_t\,(K,L)$. Similarly to the previous two constructions, the authors conjectured that $|R_\lambda(K,L)| \geq |K|^{1-\lambda}\,|L|^\lambda$ for $K,T\in\mathcal{K}_s^n$. However, after the publication of [9], Cordero-Erausquin and Klartag found a counterexample to this inequality.

We will now present a new definition for the geometric mean of two convex bodies in $\mathbb{R}^n$, which seems to satisfy some natural properties. As a first step, let us consider the geometric mean of positive numbers. Given two numbers $x,y>0$, we build two sequences by the recurrence relations

$$\begin{aligned} a_0 &= x & h_0 &= y \\ a_{n+1} &= \frac{a_n+h_n}{2} & h_{n+1} &= \left(\frac{a_n^{-1}+h_n^{-1}}{2}\right)^{-1}. \end{aligned}$$

It is an easy exercise to see that $\{a_n\}$ is decreasing, $\{h_n\}$ is increasing, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} h_n = \sqrt{xy}$.

A similar result is known to hold for positive definite matrices. Given two such matrices u and v, we define

$$\begin{aligned} A_0 &= u & H_0 &= v \\ A_{n+1} &= \frac{A_n + H_n}{2} & H_{n+1} &= \left(\frac{A_n^{-1} + H_n^{-1}}{2}\right)^{-1}. \end{aligned}$$

It is known that $\{A_n\}$ is decreasing (in the sense of matrices), $\{H_n\}$ is increasing, and the limits $\lim_{n\to\infty} A_n$ and $\lim_{n\to\infty} H_n$ exist and are equal. This joint limit is known as the geometric mean of u and v, and is often written as $u\#v$. It shares many of the properties of the geometric mean of numbers - see e.g. [15] for a survey of such properties. An explicit formula for $u\#v$ is

$$u\#v = u^{1/2}\left(u^{-1/2} v u^{-1/2}\right)^{1/2} u^{1/2},$$

but it may be better to think of $u\#v$ as the unique solution of the matrix equation $xu^{-1}x = v$

Since we already understand the arithmetic mean and harmonic mean of convex bodies, we may simply repeat the same process. For $K, T \in \mathcal{K}^n_{(0)}$ we define

$$\begin{aligned} A_0 &= K & H_0 &= T \\ A_{n+1} &= \frac{A_n + H_n}{2} & H_{n+1} &= \left(\frac{A_n^\circ + H_n^\circ}{2}\right)^\circ. \end{aligned} \tag{3.3}$$

Theorem 6. *Fix $K, T \in \mathcal{K}^n_{(0)}$ and define sequences $\{A_n\}$ and $\{H_n\}$ according to (3.3). Then $\{A_n\}$ is decreasing and $\{H_n\}$ is increasing with respect to set inclusion, and the limits $\lim_{n\to\infty} A_n$ and $\lim_{n\to\infty} H_n$ exist (in the Hausdorff sense) and are equal.*

Proof. By Theorem 2 we see that $A_n \supseteq H_n$ for every $n \geq 1$. If follows that

$$A_{n+1} = \frac{A_n + H_n}{2} \subseteq \frac{A_n + A_n}{2} = A_n,$$

and

$$H_{n+1} = \left(\frac{A_n^\circ + H_n^\circ}{2}\right)^\circ \supseteq \left(\frac{H_n^\circ + H_n^\circ}{2}\right)^\circ = H_n.$$

Hence $\{A_n\}$ is a decreasing sequence of convex bodies. It is also bounded from below by a "proper" convex body (with non-empty interior), since

$$A_n \supseteq H_n \supseteq H_1$$

for all $n \geq 1$. It follows that there exists a body $G_1 \in \mathcal{K}^n_{(0)}$ such that $A_n \to G_1$ in the Hausdorff sense. Similarly, $\{H_n\}$ is increasing and bounded from above by A_1, so it converges to some G_2.

Finally, taking the equation

$$A_{n+1} = \frac{A_n + H_n}{2}$$

and sending $n \to \infty$, we see that $G_1 = \frac{1}{2}(G_1 + G_2)$. Hence $G_1 = G_2$ and the proof is complete. □

Definition 7. The joint limit from the previous theorem is called the geometric mean of K and T:

$$G(K,T) = \lim_{n\to\infty} A_n = \lim_{n\to\infty} H_n.$$

If we need to refer to the bodies A_n and H_n from the process defining $G(K,T)$, we will write $A_n(K,T)$ and $H_n(K,T)$. It is immediate that $H_n(K,T) \subseteq G(K,T) \subseteq A_n(K,T)$ for all n. In particular, we have the arithmetic mean - geometric mean - harmonic mean inequality $H_1(K,T) \subseteq G(K,T) \subseteq A_1(K,T)$.

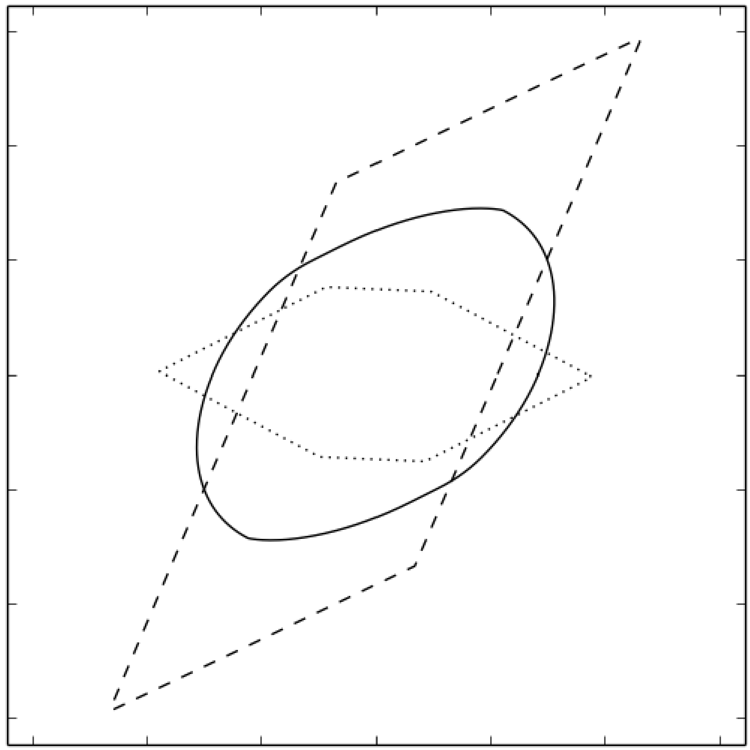

Figure 1: K, T (dashed and dotted lines) and $G(K,T)$ (solid line)

Figure 1 depicts one planar example of two convex polygons K and T and their geometric mean.

Even though our motivation is very different, the above definition was also inspired by a similar construction for 2-homogeneous functions of Asplund ([4]). We have also recently discovered a paper of Fedotov ([10]) with a similar construction.

4 Properties of the geometric mean

The following proposition summarizes some of the basic properties of the geometric mean:

Proposition 8. *1. $G(K,K) = K$.*

2. $G(K,T)$ is monotone in its arguments: If $K_1 \subseteq K_2$ and $T_1 \subseteq T_2$ then $G(K_1,T_1) \subseteq G(K_2,T_2)$.

3. $[G(K,T)]^\circ = G(K^\circ, T^\circ)$.

4. For any linear map u we have $G(uK, uT) = u(G(K,T))$.

Proof. 1. This is obvious, as $A_n(K,K) = H_n(K,K) = K$ for all $n \geq 0$.

2. If $K_1 \subseteq K_2$ and $T_1 \subseteq T_2$ then easy induction of n shows that $A_n(K_1,T_1) \subseteq A_n(K_2,T_2)$ and $H_n(K_1,T_1) \subseteq H_n(K_2,T_2)$ for all $n \geq 0$. Sending $n \to \infty$ gives the result.

3. Again, use induction to show that $[A_n(K,T)]^\circ = H_n(K^\circ, T^\circ)$ and $[H_n(K,T)]^\circ = A_n(K^\circ, T^\circ)$ for all $n \geq 0$. Send $n \to \infty$ for the result.

4. Using yet another induction, $A_n(uK, uT) = u\,(A_n(K,T))$ and $H_n(uK, uT) = u\,(H_n(K,T))$ for all $n \geq 0$. Again, we obtain the required result in the limit.

$\square$

Let us mention another easy but important property of the geometric mean: All of our means (the arithmetic mean, the harmonic mean and the geometric mean) do not depend on the choice of a scalar product on $\mathbb{R}^n$. This is obvious for the arithmetic mean, but less so for the harmonic and the geometric mean, since the polarity map $K \mapsto K^\circ$ which appears in the definition *does* depend on this choice. However, remember from the proof of Theorem 2 that

$$r_\theta\left(H(K,T)\right) = \frac{2}{\frac{1}{r_\theta(K)} + \frac{1}{r_\theta(T)}}$$

for all $\theta \in \mathbb{R}^n$, so $H(K,T)$ may be constructed from K and T without mentioning polarity or any scalar product. It follows that the harmonic mean, and hence also the geometric mean, may be defined without fixing a scalar product on our space.

Our next goal is to compute the geometric mean in several important cases, which will give us a better intuition for it.

Proposition 9. *Let K be any convex body. Then:*

1. *$G(K, K^\circ) = B_2^n$.*
2. *For any positive definite linear map u we have $G(K, uK^\circ) = u^{1/2} B_2^n$.*
3. *For any $\alpha, \beta > 0$ we have $G(\alpha K, \beta K^\circ) = \sqrt{\alpha\beta} B_2^n$.*

Proof. For (1), notice that

$$G(K,K^\circ)^\circ = G\left(K^\circ, (K^\circ)^\circ\right) = G(K^\circ, K) = G(K, K^\circ).$$

as the only body $\mathcal{K}^n_{(0)}$ to satisfy $X^\circ = X$ is $X = B_2^n$, the claim follows.

Part (2) follows from (1), as

$$G(K, uK^\circ) = u^{1/2} G\left(u^{-1/2}K, u^{1/2}K^\circ\right) = u^{1/2} G\left(u^{-1/2}K, \left(u^{-1/2}K\right)^\circ\right) = u^{1/2} B_2^n.$$

Finally, for (3) we take $u(x) = \frac{\beta}{\alpha}x$ in (2) and obtain

$$G(\alpha K, \beta K^\circ) = \alpha \cdot G\left(K, \frac{\beta}{\alpha}K^\circ\right) = \alpha \cdot \sqrt{\frac{\beta}{\alpha}} B_2^n = \sqrt{\alpha\beta} B_2^n.$$

$\square$

This proposition gives us another way to think about the geometric mean, as an extension of the notion of polarity. We would like to say that T is polar to K with respect to Z if $g(K,T) = Z$. The above proposition says that K° is indeed polar to K with respect to the Euclidean ball. Several natural problems regarding the theory of "polarity with respect to a convex body" will appear in the final section of this paper.

One may also compare this proposition with its obvious counterparts for numbers and matrices: $G(x, x^{-1}) = 1$ for every $x > 0$ and $G(u, u^{-1}) = Id$ for every positive definite matrix u. We see that the ball B_2^n plays the same role as the number 1 for positive numbers or the identity matrix for positive definite matrices. Hence it makes sense to define $\sqrt{K} = G(K, B_2^n)$. For many of the open problems discussed in this paper one may first concentrate on this special case.

Proposition 10. *Let u, v be positive-definite matrices, and let*

$$\begin{aligned}\mathcal{E}_1 &= \{x: \langle ux, x\rangle \le 1\}\\ \mathcal{E}_2 &= \{x: \langle vx, x\rangle \le 1\}\end{aligned}$$

be the corresponding ellipsoids. Then $G(\mathcal{E}_1, \mathcal{E}_2) = \{x: \langle wx, x\rangle \le 1\}$, where $w = u\#v$ is the matrix geometric mean of u and v.

Proof. We have $\mathcal{E}_1 = u^{-1/2}B_2^n$, so $\mathcal{E}_1^\circ = u^{1/2}B_2^n = \left\{x: \left\langle u^{-1}x, x\right\rangle \le 1\right\}$.

Since $v = wu^{-1}w$, we see that

$$\mathcal{E}_2 = \left\{x: \left\langle wu^{-1}wx, x\right\rangle \le 1\right\} = \left\{x: \left\langle u^{-1}wx, wx\right\rangle \le 1\right\} = \{x: wx \in \mathcal{E}_1^\circ\} = w^{-1}\mathcal{E}_1^\circ.$$

Hence by Proposition 9 we have

$$G(\mathcal{E}_1, \mathcal{E}_2) = G(\mathcal{E}_1, w^{-1}\mathcal{E}_1^\circ) = w^{-1/2}B_2^n = \{x: \langle wx, x\rangle \le 1\}$$

like we wanted. □

The above result is somewhat surprising – For ellipsoids $\mathcal{E}_1, \mathcal{E}_2$ the intermediate sets $A_n(\mathcal{E}_1, \mathcal{E}_2)$ and $H_n(\mathcal{E}_1, \mathcal{E}_2)$ are not ellipsoids. Still, in the limit we obtain that $G(\mathcal{E}_1, \mathcal{E}_2)$ is an ellipsoid. Actually $\mathcal{E}_1$ and $\mathcal{E}_2$ are dual to each other with respect to the ellipsoid $G(\mathcal{E}_1, \mathcal{E}_2)$ (i.e. if the scalar product on $\mathbb{R}^n$ is chosen in such a way that $G(\mathcal{E}_1, \mathcal{E}_2)$ is the unit ball, then $\mathcal{E}_2 = \mathcal{E}_1^\circ$).

Proposition 10 has a nice corollary regarding the Banach-Mazur distance. For symmetric convex bodies $K, T \in \mathcal{K}_s^n$ the Banach-Mazur distance $d_{BM}(K, T)$ is defined by

$$d_{BM}(K, T) = \min\{\lambda > 0: \text{ There exists a linear map } u \text{ such that } uT \subseteq K \subseteq \lambda \cdot uT\}.$$

We have the following result:

Proposition 11. *For every $K \in \mathcal{K}_s^n$ one has $d_{BM}(\sqrt{K}, B_2^n) \le \sqrt{d_{BM}(K, B_2^n)}$.*

Proof. Write $d = d_{BM}(K, B_2^n)$. By definition, there exists an ellipsoid $\mathcal{E}$ such that $\mathcal{E} \subseteq K \subseteq d \cdot \mathcal{E}$. By the monotonicity of the geometric mean it follows that $\sqrt{\mathcal{E}} \subseteq \sqrt{K} \subseteq \sqrt{d \cdot \mathcal{E}}$.

From Proposition 10 it follows that $\sqrt{\mathcal{E}} = G(\mathcal{E}, B_2^n)$ is an ellipsoid. Furthermore, the explicit formula given there immediately implies that $\sqrt{d \cdot \mathcal{E}} = \sqrt{d} \cdot \sqrt{\mathcal{E}}$. Hence we have $\sqrt{\mathcal{E}} \subseteq \sqrt{K} \subseteq \sqrt{d} \cdot \sqrt{\mathcal{E}}$ so

$$d_{BM}(\sqrt{K}, B_2^n) \le \sqrt{d} = \sqrt{d_{BM}(K, B_2^n)}.$$

□

We know from John's theorem that $d_{BM}(K, B_2^n) \le \sqrt{n}$ for all $K \in \mathcal{K}_s^n$, so we always have $d_{BM}(\sqrt{K}, B_2^n) \le n^{1/4}$. In particular, since it is known that $d_{BM}(B_p^n, B_2^n) = n^{|1/2 - 1/p|}$, it follows that there is no $K \in \mathcal{K}_s^n$ such that $\sqrt{K} = B_p^n$ if $p > 4$ or $p < 4/3$.

Finally, we conclude this section by computing an example in the plane that will be useful later:

Example 12. Fix $R > 1$ (that we will later send to ∞) and define

$$\begin{aligned} K &= [-R, R] \times \left[-\frac{1}{R}, \frac{1}{R}\right] \subseteq \mathbb{R}^2, \\ T &= \left[-\frac{1}{R}, \frac{1}{R}\right] \times [-R, R] \subseteq \mathbb{R}^2. \end{aligned}$$

Notice that $K \supseteq T^\circ$, so $G(K,T) \supseteq G(T^\circ, T) = B_2^n$. For the opposite inclusion, let us follow one iteration. Define

$$A = \frac{K+T}{2}, \qquad B = \left(\frac{K^\circ + T^\circ}{2}\right)^\circ.$$

Obviously, $A = \frac{1}{2}\left(R + \frac{1}{R}\right) B_\infty^2$. For B we use the following estimate:

$$\begin{aligned} h_{B^\circ}(x,y) &= \frac{h_{K^\circ}(x,y) + h_{T^\circ}(x,y)}{2} = \frac{1}{2}\left(\max\left(\frac{|x|}{R}, R\,|y|\right) + \max\left(R\,|x|, \frac{|y|}{R}\right)\right) \\ &\geq \frac{1}{2}\left((R\,|y|) + (R\,|x|)\right) = \frac{R}{2}\left(|x| + |y|\right). \end{aligned}$$

Since $|x| + |y| = h_{B_\infty^2}(x,y)$, we have $B^\circ \supseteq \frac{R}{2} B_\infty^2$, so $B \subseteq \frac{2}{R} B_1^2$.

Hence

$$G(K,T) = G(A,B) \subseteq G\left(\frac{1}{2}\left(R + \frac{1}{R}\right) B_\infty^2, \frac{2}{R} B_1^2\right) = \sqrt{1 + \frac{1}{R^2}}\, B_2^2,$$

where the last step follows from Proposition 9. It follows that $\lim_{R\to\infty} G(K,T) = B_2^2$.

5 Structure of the geometric mean

We now turn our attention to finer questions regarding the geometric mean. First we prove a relation between $G(A,B)$ and the logarithmic mean $L_{1/2}(A,B)$ described in Section 3:

Proposition 13. *For $K, T \in \mathcal{K}_{(0)}^n$ we have $G(K,T) \subseteq L_{1/2}(K,T)$.*

Proof. Define $\varphi_n = h\left(A_n(K,T)\right)$, and $\psi_n = h\left(H_n(K,T)\right)$, where h denotes the support function. We will also define another process by

$$\begin{aligned} &\widetilde{\varphi}_0 = h(K) && \widetilde{\psi}_0 = h(T) \\ &\widetilde{\varphi}_{n+1} = \frac{1}{2}\left(\widetilde{\varphi}_n + \widetilde{\psi}_n\right) && \widetilde{\psi}_{n+1} = \left[\frac{1}{2}\left(\widetilde{\varphi}_n^{-1} + \widetilde{\psi}_n^{-1}\right)\right]^{-1}. \end{aligned}$$

Notice that the functions $\widetilde{\varphi}_n, \widetilde{\psi}_n$ are not necessarily convex, unlike φ_n and ψ_n. Still, we claim that $\widetilde{\varphi}_n \geq \varphi_n$ and $\widetilde{\psi}_n \geq \psi_n$ for all n (here and everywhere else in the proof, inequalities between functions are meant in the pointwise sense). For $n = 0$ there is nothing to prove. If we assume the inequalities to be true for n, then for $n+1$ we hav

$$\begin{aligned} \varphi_{n+1} &= h\left(A_{n+1}\right) = h\left(\frac{A_n + H_n}{2}\right) = \frac{h\left(A_n\right) + h\left(H_n\right)}{2} = \frac{\varphi_n + \psi_n}{2} \\ &\leq \frac{\widetilde{\varphi}_n + \widetilde{\psi}_n}{2} = \widetilde{\varphi}_{n+1}. \end{aligned}$$

We also have

$$\begin{aligned}\psi_{n+1} &= h\left(H_{n+1}\right)=r\left(\frac{A_n^\circ+H_n^\circ}{2}\right)^{-1} \overset{(*)}{\leq} \left(\frac{r\left(A_n^\circ\right)+r(H_n^\circ)}{2}\right)^{-1} = \\ &= \frac{2}{h(A_n)^{-1}+h(B_n)^{-1}}=\frac{2}{\varphi_n^{-1}+\psi_n^{-1}} \leq \frac{2}{\widetilde{\varphi}_n^{-1}+\widetilde{\psi}_n^{-1}}=\widetilde{\psi}_{n+1},\end{aligned}$$

where $(*)$ was explained in the proof of Theorem 2. This completes the inductive proof.

It is a simple exercise in calculus that

$$\lim_{n\to\infty}\widetilde{\varphi}_n=\lim_{n\to\infty}\widetilde{\psi}_n=\sqrt{h(K)h(T)}.$$

Therefore, by taking the limit $n\to\infty$ in the inequality $\widetilde{\varphi}_n\geq\varphi_n$ we see that

$$h\left(G(K,T)\right)=\lim_{n\to\infty}\varphi_n\leq\lim_{n\to\infty}\widetilde{\varphi}_n=\sqrt{h(K)h(T)}.$$

This proves the result. □

For the "dual" logarithmic sum, we obtain an inclusion in the opposite direction:

Corollary 14. *For $K,T\subseteq\mathcal{K}^n_{(0)}$ and we have $G(K,T)\supseteq L_{1/2}(K^\circ,T^\circ)^\circ$.*

Proof. Applying Proposition 13 to K° and T° we see that

$$G\left(K,T\right)^\circ=G\left(K^\circ,T^\circ\right)\subseteq L_{1/2}\left(K^\circ,T^\circ\right).$$

Taking polarity, we obtain the result. □

The inclusions in the last two results may be strict. For example, take $K=B_\infty^2\subseteq\mathbb{R}^2$ and $T=B_1^2\subseteq\mathbb{R}^2$. Then $G(K,T)=B_2^2$ since $T=K^\circ$. However, a direct (yet tedious) computation shows that $L_{1/2}(K,T)$ and $L_{1/2}\left(K^\circ,T^\circ\right)^\circ$ are octagons. Figure 2 depicts the three bodies $G(K,T)$, $L_{1/2}(K,T)$ and $L_{1/2}\left(K^\circ,T^\circ\right)^\circ$.

In the figure we see that even though those three bodies are distinct, there are directions in which their radial functions coincide. This is not a coincidence, as the next proposition shows:

Proposition 15. *Let K and T be convex bodies. Assume that in direction η the bodies K and T have parallel supporting hyperplanes, with normal vector θ. Write $G=G(K,T)$. Then $h_G(\theta)=\sqrt{h_K(\theta)h_T(\theta)}$ and $r_G(\eta)=\sqrt{r_K(\eta)r_T(\eta)}$.*

Proof. Write $a=r_K(\eta)\eta\in\partial K$ and $b=r_T(\eta)\eta\in\partial T$. Since the hyperplane $\{x:\ \langle x,\theta\rangle=\langle a,\theta\rangle\}$ is a supporting hyperplane for K we know that

$$h_K(\theta)=\langle a,\theta\rangle=r_K(\eta)\cdot\langle\eta,\theta\rangle\,,$$

and similarly $h_T(\theta)=r_T(\eta)\cdot\langle\eta,\theta\rangle$.

On the one hand, by Proposition 13 we know that $h_G(\theta)\leq\sqrt{h_K(\theta)h_T(\theta)}$. On the other hand,

$$\begin{aligned}h_G(\theta) &= \sup_{\alpha\in S^{n-1}}\left(\langle\alpha,\theta\rangle\cdot r_G(\alpha)\right)\geq\langle\eta,\theta\rangle\cdot r_G(\eta) \\ &\geq \langle\eta,\theta\rangle\cdot\sqrt{r_K(\eta)r_T(\eta)}=\sqrt{h_K(\theta)h_T(\theta)},\end{aligned}$$

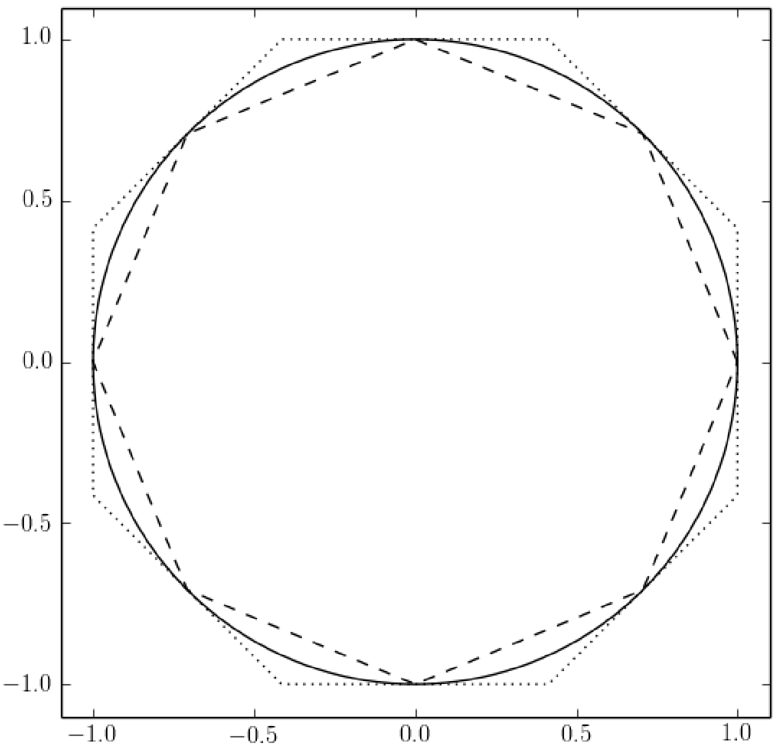

Figure 2: $G(K,T)$ (solid line), $L_{1/2}(K,T)$ (dotted line), $L_{1/2}(K^\circ,T^\circ)^\circ$ (dashed line)

where we used Corollary 14. Together we see that indeed $h_G(\theta) = \sqrt{h_K(\theta)h_T(\theta)}$. Hence we must also have $h_G(\theta) = \langle \eta, \theta \rangle \cdot r_G(\eta)$, so

$$r_G(\eta) = \frac{h_G(\theta)}{\langle \eta, \theta \rangle} = \sqrt{\frac{h_K(\theta)}{\langle \eta, \theta \rangle} \cdot \frac{h_T(\theta)}{\langle \eta, \theta \rangle}} = \sqrt{r_K(\eta) \cdot r_T(\eta)}.$$

□

Remember that if K is not smooth at a point, it may have many supporting hyperplanes at this point. The above proposition only requires that *one* supporting hyperplane of K in direction η is parallel to *one* supporting hyperplane of T in the same direction. Such directions η always exist, for any pair of convex bodies K and T. For example, if one defines a functional $\Phi : S^{n-1} \to \mathbb{R}$ by $\Phi(\eta) = \frac{r_K(\eta)}{r_T(\eta)}$, then it is enough to take the points where Φ attains its extrema.

Let us understand the relation between $G(K,T)$ and $L_{1/2}(K,T)$ in different terms. Formally we have only defined the mean $G(K,T)$ for compact sets. However, there is a natural extension of this notion to slabs: we will write $S_{\theta,c} = \{x : \ |\langle x, \theta \rangle| \le c\}$, and set

$$G\left(S_{\theta,c}, S_{\eta,d}\right) = \begin{cases} S_{\theta,\sqrt{cd}} & \text{if } \theta = \eta \\ \mathbb{R}^n & \text{otherwise.} \end{cases}$$

One way to justify this formula is to think about a slab as a degenerated ellipsoid and take a suitable limit in Proposition 10.

From this definition it is immediate that

$$L_{1/2}(K,T) = \bigcap \{G(S_{\theta,c}, S_{\eta,d}) : \ K \subseteq S_{\theta,c} \text{ and } T \subseteq S_{\eta,d}\}.$$

However, what happens if we allow arbitrary ellipsoids, and not only slabs?

Definition 16. Given convex bodies $K, T \subseteq \mathbb{R}^n$, we define the upper ellipsoidal envelope of K and T to be

$$\overline{G}(K,T) = \bigcap \{G(\mathcal{E}_1, \mathcal{E}_2) : \ K \subseteq \mathcal{E}_1 \text{ and } T \subseteq \mathcal{E}_2\},$$

Similarly we define the lower ellipsoidal envelope of K and T as

$$\underline{G}(K,T) = \overline{G}(K^\circ, T^\circ)^\circ = \operatorname{conv} \bigcup \{G(\mathcal{E}_1, \mathcal{E}_2) : \ K \supseteq \mathcal{E}_1 \text{ and } T \supseteq \mathcal{E}_2\}.$$

We obviously have $\underline{G}(K,T) \subseteq G(K,T) \subseteq \overline{G}(K,T)$. It will be interesting to know when is it true that $\overline{G}(K,T) = \underline{G}(K,T) = G(K,T)$.

To see why this may be interesting, remember that in Example 10 we proved an explicit formula for the geometric mean of ellipsoids. From this formula it is clear that

$$G(\alpha\mathcal{E}_1, \beta\mathcal{E}_2) = \sqrt{\alpha\beta} G(\mathcal{E}_1, \mathcal{E}_2)$$

for all ellipsoids $\mathcal{E}_1, \mathcal{E}_2$ and all $\alpha, \beta > 0$. Hence we also have

$$\overline{G}(\alpha K, \beta T) = \sqrt{\alpha\beta} \cdot \overline{G}(K,T),$$

and similarly for the lower envelope $\underline{G}$. It seems intuitive that this scaling property should hold for G as well, and it is obviously true whenever G coincide with one of its envelopes. However, recently Alexander Magazinov found a counterexample to this "scaling conjecture". In particular, his example shows that in general G does not have to coincide with its ellipsoidal envelopes. Magazinov's example appears as an appendix to this paper.

Another possible application of the equality $\overline{G}(K,T) = \underline{G}(K,T)$ (whenever it is true) will be given in Section (7), where we discuss a possible extension of the iteration process 3.3 to p-sums.

6 Volume inequalities

Like in the case of the logarithmic mean or the complex interpolation, it is natural to ask whether we have an inequality of the form

$$|G(K,T)| \geq \sqrt{|K|\,|T|}$$

for $K, T \in \mathcal{K}_s^n$. Such an inequality will be intimately related to the Brunn-Minkowski and log-Brunn-Minkowski inequalities (remember the discussion in Section 3 and Proposition 13), as well as the Blaschke-Santaló inequality (take $T = K^\circ$ and remember Proposition 9).

Unfortunately, we have already seen in Example 12 that this inequality is false in general. In that example we had $|K| = |T| = 4$ for all values of R, but

$$\lim_{R\to\infty} |G(K,T)| = \left|B_2^2\right| = \pi < 4.$$

Still, it seems worthwhile to understand for what classes of bodies this inequality is true, and how far it is from being true in general. For example, the following is an immediate computation:

Proposition 17. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are ellipsoids then $|G(\mathcal{E}_1, \mathcal{E}_2)| = \sqrt{|\mathcal{E}_1|\,|\mathcal{E}_2|}$.*

Proof. Using the computations and notation of Example 10 we have

$$\begin{aligned}
|G(\mathcal{E}_1, \mathcal{E}_2)| &= \left| w^{-1/2} B_2^n \right| = (\det w)^{-1/2} |B_2^n| = \det (u \# v)^{-1/2} |B_2^n| \\
&= (\det u \cdot \det v)^{-1/4} |B_2^n| = \sqrt{\left[(\det u)^{-1/2} |B_2^n| \right] \left[(\det v)^{-1/2} |B_2^n| \right]} \\
&= \sqrt{\left| u^{-1/2} B_2^n \right| \cdot \left| v^{-1/2} B_2^n \right|} = \sqrt{|\mathcal{E}_1|\,|\mathcal{E}_2|}.
\end{aligned}$$

□

Corollary 18. *The log-Brunn-Minkowski conjecture is true for ellipsoids.*

Proof. Combine the above Proposition with Proposition 13. □

In general, let us define the constant g_n to be the biggest constant such that

$$|G(K, T)| \geq g_n \cdot \sqrt{|K|\,|T|}.$$

for all $K, T \in \mathcal{K}_s^n$. Determining the asymptotics of g_n as $n \to \infty$ may have important consequences. For example, by Proposition 13 we see that

$$\left| L_{1/2}(K, T) \right| \geq g_n \sqrt{|K|\,|T|},$$

so this question is directly related to the log-Brunn-Minkowski conjecture. As another example, by monotonicity and Proposition 9 we have $G(K \cap T, K^\circ \cap T) \subseteq B_2^n \cap T$, so $|K \cap T| \cdot |K^\circ \cap T| \leq g_n^{-2} \cdot |B_2^n \cap T|^2$.

From proposition 17 and John's theorem one obtains the trivial bound $g_n \geq n^{-n/2}$. It seems possible that $g_n \geq c^n$ for some constant c. Such a result will essentially recover Klartag's result (3.2), perhaps with a different constant. It will also give a new proof of the inequality (3.1) which follows from the work of Saroglou.

7 p-additions and p-geometric means

In the introduction to this paper we took time to explain the role of the polarity map, but we took for granted the fact that the "addition" of convex bodies is indeed the Minkowski sum. However, there are other interesting additions on convex sets, such as the p-additions. This notion was introduced by Firey ([12]) and studied extensively by Lutwak ([16], [17]). For $K, T \in \mathcal{K}_{(0)}^n$ and $1 \leq p < \infty$, the p-sum $K +_p T$ is defined implicitly by the relation

$$h_\theta(K +_p T) = \left(h_\theta(K)^p + h_\theta(T)^p \right)^{1/p}.$$

The case $p = 1$ is of course the standard Minkowski addition.

Instead of the process (3.3), one may fix $1 \leq p < \infty$ and look at the following process:

$$\begin{array}{ll}
A_0 = K & H_0 = T \\
A_{n+1} = \dfrac{A_n +_p H_n}{2^{1/p}} & H_{n+1} = \left(\dfrac{A_n^\circ +_p H_n^\circ}{2^{1/p}} \right)^\circ .
\end{array} \tag{7.1}$$

(the factor $2^{1/p}$ is the correct one, since $K +_p K = 2^{1/p} K$).

Theorem 19. *Fix $K, T \in \mathcal{K}^n_{(0)}$ and $1 \le p < \infty$, and define processes $\{A_n\}$, $\{H_n\}$ by (7.1). Then $\{A_n\}$ is decreasing, $\{H_n\}$ is increasing, and the limits $\lim\limits_{n\to\infty} A_n$ and $\lim\limits_{n\to\infty} H_n$ exist (in the Hausdorff sense) and are equal.*

The proof is almost identical to the proof of Theorem 6, so we omit the details. We call this joint limit the p-geometric mean of K and T and denote it by $G_p(K,T)$.

As a side note, one may also discuss the ∞-sum of convex bodies which is the limit of p-sums as $p \to \infty$. Explicitly

$$h_\theta(K +_\infty T) = \lim_{p\to\infty} \left(h_\theta(K)^p + h_\theta(T)^p\right)^{1/p} = \max\left\{h_\theta(K), h_\theta(T)\right\},$$

so $K +_\infty T = K \vee T$, the convex hull of the union $K \cup T$. For $p = \infty$ the process (7.1) becomes

$$\begin{array}{ll} A_0 = K & H_0 = T \\ A_{n+1} = A_n \vee H_n & H_{n+1} = A_n \cap H_n, \end{array} \tag{7.2}$$

but this process does not converge unless $K = T$. Indeed, for every $n \ge 1$ we have $A_n = K \vee T$ and $H_n = K \cap T$. Hence we will only discuss $1 \le p < \infty$.

All the results of this paper remain true when $G(K,T)$ is replaced by $G_p(K,T)$, with almost identical proofs. In particular $G_p(K, K^\circ) = B_2^n = G(K, K^\circ)$ for all $K \in \mathcal{K}^n_s$, and $G_p(\mathcal{E}_1, \mathcal{E}_2) = G(\mathcal{E}_1, \mathcal{E}_2)$. In fact, we have computed a few examples using a computer and did not find an example where $G_p(K,T) \ne G_q(K,T)$. Is it possible that $G_p(K,T)$ does not depend on p, at least on some non-trivial cases?

In this direction it is worth mentioning that since for ellipsoids $G_p(\mathcal{E}_1, \mathcal{E}_2)$ does not depend on p, the ellipsoidal envelopes $\overline{G}(K,T)$ and $\underline{G}(K,T)$ from Definition 16 also do not depend on p. In particular, we have $\underline{G}(K,T) \subseteq G_p(K,T) \subseteq \overline{G}(K,T)$ for all $1 \le p < \infty$. From here we see that if for some bodies $K, T \in \mathcal{K}^n_s$ we have $\underline{G}(K,T) = \overline{G}(K,T)$, then $G_p(K,T)$ is indeed independent of p. As discussed in Section 5, we do not know when this is the case.

8 Functional constructions

So far we only discussed constructions of new convex bodies out of old ones. However, similar constructions can be used for convex functions as well. We denote by $\mathrm{Cvx}(\mathbb{R}^n)$ the class of all convex and lower semi-continuous functions $\varphi : \mathbb{R}^n \to (-\infty, \infty]$. The addition on $\mathrm{Cvx}(\mathbb{R}^n)$ is the regular pointwise addition, and the order is the pointwise order ($\varphi \le \psi$ if $\varphi(x) \le \psi(x)$ for all x). Like $\mathcal{K}^n_{(0)}$, the class $\mathrm{Cvx}(\mathbb{R}^n)$ also has an essentially unique duality, the Legendre transform

$$\varphi^*(y) = \sup_{x\in\mathbb{R}^n} \left[\langle x, y\rangle - \varphi(x)\right].$$

More formally, we have the following theorem of Artstein-Avidan and Milman ([1], [3]):

Theorem 20. *Every order reversing involution $\mathcal{T} : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ is the Legendre transform up to linear terms.*

Explicitly, there exists a constant $C \in \mathbb{R}$, a vector $v \in \mathbb{R}^n$, and an invertible symmetric linear transformation $B \in GL(n)$ such that

$$\left(\mathcal{T}\varphi\right)(x) = \varphi^*\left(Bx + v\right) + \langle x, v\rangle + C.$$

Hence we may think of φ^* as the inverse "φ^{-1}" and attempt to repeat the constructions of the previous sections. Notice that for functions the harmonic mean $H(\varphi, \psi) = \left[\frac{1}{2}\left(\varphi^* + \psi^*\right)\right]^*$ is exactly the inf-convolution, used by Asplund in [4]:

$$\left(H(\varphi,\psi)\right)(x) = \frac{1}{2}\inf_{y\in\mathbb{R}^n}\left(\varphi(x+y) + \psi(x-y)\right).$$

As a recent example of the benefits of thinking of φ^* as φ^{-1}, Rotem recently proved the following result ([20]):

Theorem 21. *For every $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ one has*

$$(\varphi+\delta)^* + (\varphi^*+\delta)^* = \delta,$$

where $\delta(x) = \frac{1}{2}|x|^2$ and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$.

Notice that this theorem is the analogue for convex functions of the trivial identity $\frac{1}{x+1} + \frac{1}{1/x+1} = 1$ for positive real numbers. The function δ plays the role of the number 1 as δ is the unique function satisfying $\delta^* = \delta$. This theorem has applications for functional Blaschke-Santaló type inequalities and for the theory of summands.

By fixing a convex body $K \in \mathcal{K}^n_{(0)}$ and choosing $\varphi = \frac{1}{2}h_K^2$ in Theorem 21, it was shown in [20] that

$$(K +_2 B_2^n)^\circ +_2 (K^\circ +_2 B_2^n)^\circ = B_2^n$$

where the 2-sum $+_2$ was defined in the previous section. However, it turns out that the 2-sum cannot be replaced by the Minkowski sum, as the identity

$$(K + B_2^n)^\circ + (K^\circ + B_2^n)^\circ = B_2^n$$

is simply false. So in this example not only the theory can be extended to the functional case, but the functional case is better behaved than the classical case of convex bodies.

The theory of continued fractions can also be extended to this functional case. Specifically, Molchanov proves the following theorem in [18]:

Theorem 22. *Let $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ be a non-negative function with $\varphi(0) = 0$. Assume that $\frac{r}{2}|x|^2 \le \varphi(x) \le \frac{R}{2}|x|^2$ for all $x \in \mathbb{R}^n$, for some constants r, R that satisfy $r^2 + 4\frac{r}{R} > 4$. Then the continued fraction $[\varphi, \varphi, \varphi, \ldots]$ converges to a function $\zeta \in \mathrm{Cvx}(\mathbb{R}^n)$, and ζ solves the functional equation $\zeta^* = \zeta + \varphi$.*

The convergence of $[\varphi, \varphi, \varphi, \ldots]$ is proved with respect to the metric

$$d(\varphi, \psi) = \min\{r > 0 : \; f \le g + r\delta \text{ and } g \le f + r\delta\}.$$

We will not give the details of the proof.

Finally, the construction of the geometric mean may also be carried out for convex functions. Given $\varphi, \psi \in \mathrm{Cvx}(\mathbb{R}^n)$ we define

$$\begin{aligned} \alpha_0 &= \varphi & \eta_0 &= \psi \\ \alpha_{n+1} &= \frac{\alpha_n + \eta_n}{2} & \eta_{n+1} &= \left(\frac{\alpha_n^* + \eta_n^*}{2}\right)^*. \end{aligned}$$

It is then possible to prove the following result:

Theorem 23. *Assume $\varphi, \psi \in \mathrm{Cvx}(\mathbb{R}^n)$ are everywhere finite. Then the pointwise limit*

$$\rho = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n$$

exists. We call ρ the geometric mean of φ and ψ and write $\rho = G(\varphi, \psi)$. Furthermore, the functional geometric mean has the following properties:

1. *$G(\varphi, \varphi) = \varphi$.*

2. *$G(\varphi, \psi)$ is monotone in its arguments: If $\varphi_1 \subseteq \varphi_2$ and $\psi_1 \subseteq \psi_2$ then $G(\varphi_1, \psi_1) \subseteq G(\varphi_2, \psi_2)$.*
3. *$[G(\varphi, \psi)]^* = G(\varphi^*, \psi^*)$.*
4. *$G(\varphi, \varphi^*) = \delta$.*
5. *For any linear map u we have $G(\varphi \circ u, \psi \circ u) = G(\varphi, \psi) \circ u$.*

We omit the details of the proof, as it is very similar to Theorem 6, Proposition 8 and Proposition 9.

Finally, let us note that in some ways the geometric mean of convex functions is even more well behaved than the geometric mean of convex bodies. For example, the following theorem is proved in [20]:

Theorem 24. *The geometric mean of convex functions is concave in its arguments. More explicitly, fix $\varphi_0, \varphi_1, \psi_0, \psi_1 \in \mathrm{Cvx}(\mathbb{R}^n)$ and $0 < \lambda < 1$. Define $\varphi_\lambda = (1-\lambda)\varphi_0 + \lambda\varphi_1$ and $\psi_\lambda = (1-\lambda)\psi_0 + \lambda\psi_1$. Then*

$$G(\varphi_\lambda, \psi_\lambda) \supseteq (1-\lambda) \cdot G(\varphi_0, \psi_0) + \lambda G(\varphi_1, \psi_1)$$

whenever all the geometric means in this expression are well defined.

This theorem is the natural extension of the fact that $f(x, y) = \sqrt{xy}$ is a concave function on $(\mathbb{R}_+)^2$. Like in Theorem 21, the functional version immediately implies a theorem for convex bodies *with the* 2*-sum*: For every convex bodies $K_0, K_1, T_0, T_1 \in \mathcal{K}^n_{(0)}$ one has

$$G_2(K_\lambda, T_\lambda) \supseteq \sqrt{1-\lambda} G_2(K_0, T_0) +_2 \sqrt{\lambda} G_2(K_1, T_1),$$

where $K_\lambda = \sqrt{1-\lambda} K_0 +_2 \sqrt{\lambda} K_1$ and $T_\lambda = \sqrt{1-\lambda} T_0 +_2 \sqrt{\lambda} T_1$.

However, it is also proved in [20] that the concavity property does not hold for the geometric mean of convex bodies with the regular Minkowski addition. So, like in Theorem 21, the functional theory is better behaved than the classical theory.

9 Open problems

We conclude this paper by clearly listing the open problems that were mentioned in the previous sections, together with a few other.

1. As explained in Section 4, we would like to think of the relation $G(K, T) = Z$ as "T is polar to K with respect to Z". This ideology raises the following questions:

   (a) **Domain of polarity**: Fix $Z \in \mathcal{K}^n_{(0)}$. For which convex bodies K there exists a T such that $G(K, T) = Z$? In other words, what is the natural domain of this "extended polarity"?
   As a particular sub-problem, assume that for every $K \in \mathcal{K}^n_{(0)}$ there exists a $T \in \mathcal{K}^n_{(0)}$ such that $G(K, T) = Z$. Does it follow that Z is an ellipsoid?

   (b) **Uniqueness**: Is the polar body to K with respect to Z always unique? More explicitly, if $K, T_1, T_2 \in \mathcal{K}^n_{(0)}$ satisfy $G(K, T_1) = G(K, T_2)$, does it follow that $T_1 = T_2$?

   (c) **Order reversing**: Assume that $G(K_1, T_1) = G(K_2, T_2)$ and $K_1 \supseteq K_2$. Does it follow that $T_1 \subseteq T_2$? In other words, is the "extended polarity" transform order reversing? Notice that an affirmative answer to this question implies an affirmative answer to the previous question. It is also worth mentioning that $G(K, T_1) \supseteq G(K, T_2)$ does *not* imply that $T_1 \supseteq T_2$, and one can even construct a counterexample where K, T_1, T_2 are ellipsoids.

2. For which bodies $K, T \in \mathcal{K}_s^n$ we have $\overline{G}(K,T) = \underline{G}(K,T)$?

   Remember from the discussion in Sections 5 and 7 that this question is related to the next two. Also remember that by Magazinov's example in the appendix, the answer to this question is not "always".

3. Fix $K, T \in \mathcal{K}_{(0)}^n$ and $\alpha, \beta > 0$. When is it true that $G(\alpha K, \beta T) = \sqrt{\alpha\beta} G(K,T)$?

   Since this equality holds when $\alpha = \beta$, it is enough to assume that $\beta = 1$ or that $\alpha = \frac{1}{\beta}$. Again, the answer to this question is not "always".

4. Fix $K, T \in \mathcal{K}_{(0)}^n$. When is $G_p(K,T)$ as defined in Section 7 independent on $p \in [1, \infty)$?

5. As discussed in Section 6, what is the asymptotic behavior of

$$g_n = \inf_{K,T \in \mathcal{K}_s^n} \frac{|G(K,T)|}{\sqrt{|K|\,|T|}}$$

   as $n \to \infty$?

6. Does there exists an "exponential map" $E : \mathcal{K}_{(0)}^n \to \mathcal{K}_{(0)}^n$ such that

$$E\left(\frac{K+T}{2}\right) = G\left(E(K), E(T)\right)?$$

   What should be the image $\mathcal{I} \subseteq \mathcal{K}_{(0)}^n$ of E? Since $\exp\left([0,\infty)\right) = [1,\infty)$, it may be possible to take $\mathcal{I} = \left\{K \in \mathcal{K}_{(0)}^n : \; K \supseteq B_2^n\right\}$.

   It may be easier to construct the "logarithm" $L : \mathcal{I} \to \mathcal{K}_{(0)}^n$ with the property

$$G(L(K), L(T)) = \frac{L(K) + L(T)}{2}.$$

7. How to properly define the weighted geometric mean of two convex bodies?

   For numbers $x, y > 0$ and $\lambda \in [0,1]$, the λ-geometric mean of x and y is simply $x^{1-\lambda} y^{\lambda}$. For positive-definite matrices, the λ-geometric mean of u and v is usually defined as

$$u \#_\lambda v = u^{1/2} \left(u^{-1/2} v u^{-1/2}\right)^{\lambda} u^{1/2}.$$

   What should the definition be for convex bodies? It is possible to define for example $G_{1/4}(K,T) = G\left(G(K,T), T\right)$ and so on, but is there a more direct approach?

References

[1] Shiri Artstein-Avidan and Vitali Milman. A characterization of the concept of duality. *Electronic Research Announcements in Mathematical Sciences*, 14:42–59, 2007.

[2] Shiri Artstein-Avidan and Vitali Milman. The concept of duality for measure projections of convex bodies. *Journal of Functional Analysis*, 254(10):2648–2666, may 2008.

[3] Shiri Artstein-Avidan and Vitali Milman. The concept of duality in convex analysis, and the characterization of the Legendre transform. *Annals of Mathematics*, 169(2):661–674, mar 2009.

[4] Edgar Asplund. Averaged norms. *Israel Journal of Mathematics*, 5(4):227–233, oct 1967.

[5] Károly J. Böröczky, Erwin Lutwak, Deane Yang, and Gaoyong Zhang. The log-Brunn-Minkowski inequality. *Advances in Mathematics*, 231(3-4):1974–1997, oct 2012.

[6] Károly J. Böröczky and Rolf Schneider. A characterization of the duality mapping for convex bodies. *Geometric and Functional Analysis*, 18(3):657–667, aug 2008.

[7] Jean Bourgain and Vitali Milman. New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$. *Inventiones Mathematicae*, 88(2):319–340, jun 1987.

[8] Dario Cordero-Erausquin. Santaló's inequality on $\mathbb{C}^n$ by complex interpolation. *Comptes Rendus Mathematique*, 334(9):767–772, jan 2002.

[9] Dario Cordero-Erausquin and Bo'az Klartag. Interpolations , Convexity and Geometric Inequalities. In Bo'az Klartag, Shahar Mendelson, and Vitali Milman, editors, *Geometric Aspects of Functional Analysis, Israel Seminar 2006-2010*, volume 2050 of *Lecture Notes in Mathematics*, pages 151–168. Springer, Berlin, Heidelberg, 2012.

[10] V. P. Fedotov. Geometric mean of convex sets. *Journal of Soviet Mathematics*, 10(3):488–491, sep 1978.

[11] William J. Firey. Polar means of convex bodies and a dual to the Brunn-Minkowski theorem. *Canadian Journal of Mathematics*, 13:444–453, 1961.

[12] William J. Firey. p-means of convex bodies. *Mathematica Scandinavica*, 10:17–24, 1962.

[13] Peter M. Gruber. The endomorphisms of the lattice of norms in finite dimensions. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 62(1):179–189, 1992.

[14] Bo'az Klartag. Marginals of Geometric Inequalities. In Vitali Milman and Gideon Schechtman, editors, *Geometric Aspects of Functional Analysis, Israel Seminar 2004-2005*, volume 1910 of *Lecture Notes in Mathematics*, pages 133–166. Springer, Berlin, Heidelberg, 2007.

[15] Jimmie Lawson and Yongdo Lim. The Geometric Mean , Matrices , Metrics , and More. *The American Mathematical Monthly*, 108(9):797–812, 2001.

[16] Erwin Lutwak. The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem. *Journal of Differential Geometry*, 38(1):131–150, 1993.

[17] Erwin Lutwak. The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas. *Advances in Mathematics*, 118(2):244–294, 1996.

[18] Ilya Molchanov. Continued fractions built from convex sets and convex functions. *Communications in Contemporary Mathematics*, oct 2014.

[19] Gilles Pisier. *The volume of convex bodies and Banach space geometry*, volume 94 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[20] Liran Rotem. Algebraically inspired results on convex functions and bodies. *Submitted*.

[21] Liran Rotem. A letter: The log-Brunn-Minkowski inequality for complex bodies. *arXiv:1412.5321*, dec 2014.

[22] Alexander Rubinov. *Abstract Convexity and Global Optimization*, volume 44 of *Nonconvex Optimization and Its Applications*. Springer US, Boston, MA, 2000.

[23] Christos Saroglou. More on logarithmic sums of convex bodies. *arXiv:1409.4346*, pages 1–23, sep 2014.

[24] Christos Saroglou. Remarks on the conjectured log-Brunn-Minkowski inequality. *Geometriae Dedicata*, jul 2014.

[25] Stephen Semmes. Interpolation of Banach Spaces, Differential Geometry and Differential Equations. *Revista Matemática Iberoamericana*, 4(1):155–176, 1988.

Appendix: A counterexample to the scaling property of the geometric mean

Alexander Magazinov

Theorem A. *Let K be a square with vertices $(\pm 4, 0)$ and $(0, \pm 4)$. Let T be a hexagon with vertices $(0, \pm 4)$, $(\pm 2, \pm 1)$ (the signs in the last expression are taken independently). Then*

$$G(K,T) \neq G\left((1+\varepsilon)K, \frac{1}{1+\varepsilon}T\right),$$

if $\varepsilon \neq 0$ and $|\varepsilon|$ is small enough.

The following lemma is the key to Theorem A. By a non-reflex angle in $\mathbb{R}^2$ we mean a closed convex cone $C \subseteq \mathbb{R}^2$ with vertex at $(0,0)$ which has a non-empty interior, but that does not contain a full line. The notation $[u_1, u_2]$ for $u_1, u_2 \in \mathbb{R}^2$ denotes the closed interval with endpoints u_1 and u_2:

Lemma B. *Let $C \subseteq \mathbb{R}^2$ be a non-relfex angle. Fix $K, T \in \mathcal{K}^2_{(0)}$ such that*

$$C \cap \partial K = [u_1, u_2], \quad C \cap \partial T = [\alpha u_1, \beta u_2]$$

for some $u_1, u_2 \in \mathbb{R}^2$, where $0 < \alpha < \beta$. Assume that the lines

$$\ell = \{u_1 + t(\alpha u_1 - \beta u_2) : \ t \in \mathbb{R}\}, \qquad \ell' = \{\beta u_2 + t(u_1 - u_2) : \ t \in \mathbb{R}\}$$

are support lines to K and T respectively. Then

1. *There exists a unique Euclidean scalar product $Q(\cdot,\cdot)$ in $\mathbb{R}^2$ such that*
$$Q(\alpha u_1 - \beta u_2, u_1) = Q(u_1 - u_2, u_2) = 0, \quad Q(u_1, \alpha u_1) = 1. \tag{A.1}$$
2. *If Q is as above, then the curvilinear segment $\partial G(K,T) \cap C$ is an arc of the ellipse*
$$\{v \in \mathbb{R}^2 : Q(v,v) = 1\}.$$

Outline of the proof. Fix a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with the following properties:

$$|f(u_1)| = 1/\sqrt{\alpha}, \quad |f(u_2)| = 1/\sqrt{\beta}, \quad \angle(f(u_1), f(u_2)) = \arccos\sqrt{\alpha/\beta},$$

where $\angle(v,w)$ denotes the angle between the vectors v and w. For the existence in assertion 1 it is enough to set

$$Q(v,v') = \langle f(v), f(v') \rangle . \tag{A.2}$$

Uniqueness is immediate since (A.1) gives three linearly independent linear equations in the three variables $Q(u_1, u_1)$, $Q(u_1, u_2)$, $Q(u_2, u_2)$.

Now we prove assertion 2, keeping f from above. Note that the segment $[f(u_1), f(u_2)]$ is orthogonal to the vector $f(u_2)$, and the segment $[f(\alpha u_1), f(\beta u_2)]$ is orthogonal to the vector $f(u_1)$.

By construction $\alpha|f(u_1)|^2 = 1$. Then, since the triangles $(\mathbf{0}, f(u_1), f(u_2))$ and $(\mathbf{0}, \alpha f(u_1), \beta f(u_2))$ are both right-angled and have the same angle at $(0,0)$ they are similar, so we have $\beta|f(u_2)|^2 = 1$.

We say that a ray R from the origin is an *orthogonality ray* for a convex body $X \in \mathcal{K}^2_{(0)}$ if a line through the point $R \cap \partial X$ in the direction orthogonal to R is a support line to X. The condition that ℓ and ℓ' are support lines to K and T implies that the two rays

$$R_i = \{tf(u_i) : \ t > 0\} \quad (i = 1, 2)$$

are orthogonality rays for both $f(K)$ and $f(T)$.

Note that

$$f(K)^\circ \cap f(C) = f(T) \cap f(C),$$

and the rays R_1 and R_2 are orthogonality rays for the body $f(K)^\circ$. Now we claim that

$$A_i(f(K), f(T)) \cap f(C) = A_i(f(K), f(K)^\circ) \cap f(C),$$

$$H_i(f(K), f(T)) \cap f(C) = H_i(f(K), f(K)^\circ) \cap f(C),$$

and R_1 and R_2 are orthogonality rays for each of the bodies $A_i(f(K), f(T))$, $H_i(f(K), f(T))$, $A_i(f(K), f(K)^\circ)$, $H_i(f(K), f(K)^\circ)$. Indeed, this can be checked straightforwardly by induction over i.

Passing to the limit, we have $G(f(K), f(T)) \cap f(C) = B_2^2 \cap f(C)$, so by Proposition 8 we have

$$G(K, T) \cap C = f^{-1}(B_2^2) \cap C.$$

Hence $\partial G(K, T) \cap C$ is indeed an arc of an ellipse $Q(v, v) = 1$, and assertion 2 is proved. $\square$

Proof of Theorem A. We will prove the following claim. Let ε_1 and ε_2 be arbitrary real numbers such that $\max(|\varepsilon_1|, |\varepsilon_2|)$ is small enough. Let

$$C = \{(x, y) : x < 12y < 4x\}$$

be an open angle. We prove that the curve $\partial G\left((1 + \varepsilon_1)K, (1 + \varepsilon_2)T\right) \cap C$ contains exactly one non-smooth point, which lies on the line

$$\frac{x}{6 + 4\varepsilon_1 + 2\varepsilon_2} = \frac{y}{1 + \varepsilon_2},$$

pointing to the vertex $(3 + 2\varepsilon_1 + \varepsilon_2, (1 + \varepsilon_2)/2)$ of the body $\frac{1}{2}((1 + \varepsilon_1)K + (1 + \varepsilon_2)T)$.

We will give the proof for $\varepsilon_1 = \varepsilon_2 = 0$, as one can check that the argument is applicable in the general case.

Consider the angles

$$C_1 = \{(x, y) : 0 \le 6y \le x\},$$

$$C_2 = \{(x, y) : x \le 6y \le 3x\}.$$

We claim that for $i = 1, 2$ the curvilinear segments $\partial G(K, T) \cap C_i$ are elliptic arcs, and these arcs arise from distinct ellipses.

We have $G(K, T) = G(K_1, T_1)$, where

$$K_1 = \frac{K + T}{2}, \qquad T_1 = \left(\frac{K^\circ + T^\circ}{2}\right)^\circ.$$

In the positive quadrant the vertices of K_1 are

$$u_1 = (3, 0), \quad u_2 = (3, 1/2), \quad u_3 = (2, 2), \quad u_4 = (0, 4),$$

and the vertices of T_1 in the positive quadrant are

$$v_1 = (8/3, 0), \quad v_2 = (16/7, 8/7), \quad v_3 = (0, 4).$$

Clearly, Lemma B applies to K_1 , T_1 and each C_i. Hence $\partial G(K,T) \cap C_i$ are indeed elliptic arcs.

Assume these arcs belong to the same ellipse $Q(v, v) = 1$. Then

$$Q(v_1, u_1 - u_2) = Q(v_2, u_2 - u_3) = Q(u_2, v_1 - v_2) = 0.$$

But if

$$Q((x_1, y_1), (x_2, y_2)) = ax_1x_2 + b(x_1y_2 + x_2y_1) + cy_1y_2,$$

this would yield $a = b = c = 0$, a contradiction. Therefore the common point of the arcs $\partial G(K,T) \cap C_i$ is a non-smooth point of $\partial G(K, T)$.

Consequently, the only non-smooth point of the curve $\partial G((1 + \varepsilon_1)K, (1 + \varepsilon_2)T) \cap C$ changes its angular direction from the origin, even under the additional assumption $(1 + \varepsilon_1)(1 + \varepsilon_2) = 1$. This immediately implies

$$G(K, T) \neq G\left((1 + \varepsilon_1)K, \frac{1}{1 + \varepsilon_1}T\right).$$

□

Remark. Nevertheless, the identity

$$G(K, T) = G(aK, a^{-1}T) \tag{A.3}$$

holds for some wide class of two-dimensional bodies. For instance, let K_0 be a regular n-gon and $T_0 = K_0^\circ$. Consider convex n-gons K and T that are obtained by arbitrary small enough perturbations of the vertices of K_0 and T_0 respectively. Then Lemma B allows one to reconstruct $\partial G(aK, a^{-1}T)$ completely and thus verify that (A.3) holds true for such K and T.