"Irrational" constructions in Convex Geometry

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1 Introduction

We begin by quickly recalling some basic definitions from convex geometry. We refer the reader to [20] for more information. We will denote by $\mathcal{K}_s^n$ the class of origin-symmetric convex bodies in $\mathbb{R}^n$, i.e. convex sets $K \subseteq \mathbb{R}^n$ such that K is compact, has non-empty interior, and $K = -K$. While some of the definitions given in this note make sense for non-symmetric bodies, symmetry will be important for the main results.

For a convex $K \in \mathcal{K}_s^n$ its support function $h_K : \mathbb{R}^n \to [0, \infty)$ is defined by

$$h_K(y) = \max_{x \in K} \langle x, y \rangle .$$

The function h_K uniquely defines the body K. The Minkowski sum $K + T$ of two convex bodies K and T is defined by

$$K + T = \{x + y : \; x \in K, \; y \in T\} .$$

Similarly, for $\lambda > 0$ we define $\lambda K = \{\lambda x : \; x \in K\}$. The Minkowski sum and the support function are related by the identity $h_{\lambda K + T} = \lambda h_K + h_T$.

Ellipsoids will play a special role in this note. For us an ellipsoid will always mean a centered ellipsoid, which is a linear image of the Euclidean ball

$$B_2^n = \{x : \; |x| \le 1\} .$$

For every ellipsoid $\mathcal{E}$ there exists a unique positive definite linear map $u_{\mathcal{E}}$ such that $h_{\mathcal{E}}(y) = \sqrt{\langle u_{\mathcal{E}} y, y \rangle}$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^n$. Conversely, for every positive definite map u the function $h(y) = \sqrt{\langle uy, y \rangle}$ is the support function of some ellipsoid. It follows that one may identify the class of ellipsoids in $\mathbb{R}^n$ with the class of all $n \times n$ positive definite matrices. However, let us warn the reader that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are ellipsoids then $\mathcal{E}_1 + \mathcal{E}_2$ is usually

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Research supported in part by ISF grant 826/13 and BSF grant 2012111

not an ellipsoid. The sum of two ellipsoids is an ellipsoid if one replaces the standard notion of Minkowski addition with the 2-addition, as defined by Firey ([9]) and studied extensively by Lutwak and others (see, e.g. [12], [13]). We will not need 2-additions in this note.

For a body K, the support function h_K is a norm on $\mathbb{R}^n$. Its unit ball is the polar body of K, which is denoted by K°:

$$K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in K\}.$$

The polarity map $K \mapsto K^\circ$ can be characterized as the unique order reversing involution on $\mathbb{R}^n$. To be exact, we have the following result:

Theorem 1. *Fix $n > 1$, and let $\mathcal{T} : \mathcal{K}_s^n \to \mathcal{K}_s^n$ be a map such that:*

- *$\mathcal{T}$ is an involution: $\mathcal{T}\mathcal{T}K = K$ for every $K \in \mathcal{K}_s^n$.*
- *$\mathcal{T}$ is order reversing: If $K_1 \subseteq K_2$ for some $K_1, K_2 \in \mathcal{K}_s^n$ then $\mathcal{T}K_1 \supseteq \mathcal{T}K_2$.*

Then there exists an invertible symmetric linear map u such that $\mathcal{T}K = uK^\circ$.

For the class $\mathcal{K}_s^n$, this theorem follows from a result of Gruber ([10]). Similar results on different classes of convex bodies were proved by Artstein-Avidan and Milman ([1]) and by Böröczky and Schneider ([6]).

The class of convex functions is also equipped with a unique order reversing involution. More formally, let $\mathrm{Cvx}(\mathbb{R}^n)$ denote the class of lower semi-continuous convex functions $\varphi : \mathbb{R}^n \to (-\infty, \infty]$. The Legendre transform, mapping a function φ to

$$\varphi^*(y) = \sup_{x \in \mathbb{R}^n} (\varphi(x) - \langle x, y\rangle)$$

is an order reversing involution. As was shown by Artstein-Avidan and Milman in [2], it is essentially the only such transform:

Theorem 2. *Fix $n > 1$, and let $\mathcal{T} : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ be a map such that:*

- *$\mathcal{T}$ is an involution: $\mathcal{T}(\mathcal{T}\varphi) = \varphi$ for all $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$.*
- *$\mathcal{T}$ is order reversing: If $\varphi_1 \leq \varphi_2$ then $\mathcal{T}\varphi_1 \geq \mathcal{T}\varphi_2$.*

Then there exists a constant $C \in \mathbb{R}$, a vector $v \in \mathbb{R}^n$, and an invertible symmetric linear map u such that

$$(\mathcal{T}\varphi)(x) = \varphi^*(ux + v) + \langle x, v\rangle + C.$$

On the set of positive numbers there is also a very natural order reversing bijection – the inversion map $x \to x^{-1}$. Because of this similarity between the polarity map, the Legendre transform and the inversion map, we would like to think about the polar body K° and the Legendre transform φ^* as the inverses "K^{-1}" and "φ^{-1}". Under this interpretation, most constructions we know in convexity are "rational constructions" – built by a finite number of additions and "inversions". It appears that the time has come for "irrational constructions" as well. For example, in [15] Molchanov uses this ideology to build continued fractions of convex bodies and convex functions. In particular, if $K \supseteq B_2^n$ is a compact convex body then the process

$$\left(K + (K + (K + \cdots)^\circ)^\circ\right)^\circ$$

converges to a limit Z. This Z is the unique solution of the "quadratic equation" $Z^\circ = Z + K$. More generally, one may also consider periodic continued fractions with period> 1 to be solutions of more generalized quadratic equations.

Another paper in this direction is [19], where a surprising identity for convex functions is proved using the same ideology.

2 Geometric mean of convex bodies; Ellipsoidal version

For every ellipsoid $\mathcal{E}$ we have $u_{\mathcal{E}^\circ} = (u_{\mathcal{E}})^{-1}$. In other words, the polarity operation on the class of ellipsoids corresponds to the inversion $u \mapsto u^{-1}$ on the class of positive definite matrices. The inversion map is also an order-reversing involution, when the order is the standard matrix order: $u_1 \succcurlyeq u_2$ if $u_1 - u_2$ is positive semidefinite.

For two positive definite matrices u and v, their arithmetic mean is of course $\frac{u+v}{2}$ and their harmonic mean is $\left(\frac{u^{-1}+v^{-1}}{2}\right)^{-1}$. The geometric mean of such matrices is more difficult to define: if u and v commute then uv is positive definite and one may simply consider $(uv)^{\frac{1}{2}}$, but if the matrices do not commute then this square root is not well defined. However, there exists a useful notion of such a geometric mean, which was first discovered by Pusz and Woronowicz in [17]. An explicit formula for the geometric mean of u and v is

$$u\#v = u^{\frac{1}{2}} \left(u^{-\frac{1}{2}} v u^{-\frac{1}{2}}\right)^{\frac{1}{2}} u^{\frac{1}{2}}. \tag{2.1}$$

This formula does not have an obvious analogue for convex bodies because we do not know what is means to "multiply" two such bodies. However, it turns out that there is another way to construct $u\#v$. If one defines two sequences $\{u_m\}_{m=0}^\infty$ and $\{v_m\}_{m=0}^\infty$ by

$$\begin{aligned} u_0 &= u & v_0 &= v \\ u_{m+1} &= \frac{u_m + v_m}{2} & v_{m+1} &= \left(\frac{u_m^{-1} + v_m^{-1}}{2}\right)^{-1}, \end{aligned}$$

then $\lim_{m\to\infty} u_m = \lim_{m\to\infty} v_m = u\#v$. The reader may consult [11] for a survey on the matrix geometric mean, including a proof that this definition is equivalent to the previous one.

Based on the analogy between polarity and inversion, one may make the following definition (see [14]) :

Definition 3. Fix convex bodies $K, T \in \mathcal{K}_s^n$, define two sequences $\{A_m\}_{m=0}^\infty$ and $\{H_m\}_{m=0}^\infty$ by

$$\begin{aligned} A_0 &= K & H_0 &= T \\ A_{m+1} &= \frac{A_m + H_m}{2} & H_{m+1} &= \left(\frac{A_m^{-1} + H_m^{-1}}{2}\right)^{-1}. \end{aligned}$$

The geometric mean of K and T is

$$g(K,T) = \lim_{m\to\infty} A_m = \lim_{m\to\infty} H_m.$$

A proof that these limits exist (in the Hausdorff sense) and are equal to each other appears in [14]. A very similar construction for 2-homogeneous functions was carried out by Asplund in [3], for very different reasons. His paper inspired Milman, who had a talk on the subject in Vulich Seminar (in 70/71). One of the participants in this seminar was Fedotov, who later published a short paper on the subject ([8]).

The geometric mean has many desirable properties, which are summarized in the following proposition:

Proposition 4 ([14])**.** *1. $g(K,K) = K$.*

2. *g is symmetric in its arguments: $g(K,T) = g(T,K)$.*
3. *g is monotone in its arguments: If $K_1 \subseteq K_2$ and $T_1 \subseteq T_2$ then $g(K_1,T_1) \subseteq g(K_2,T_2)$.*
4. *g satisfies the harmonic mean – geometric mean – arithmetic mean inequality*

$$\left(\frac{K^\circ + T^\circ}{2}\right)^\circ \subseteq g(K,T) \subseteq \frac{K+T}{2}.$$

5. *$[g(K,T)]^\circ = g\left(K^\circ, T^\circ\right)$.*
6. *$g(K,K^\circ) = B_2^n$.*
7. *For any linear map u we have $g(uK,uT) = u\left(g(K,T)\right)$. In particular $g(\lambda K, \lambda T) = \lambda g(K,T)$.*

It turns out that the geometric mean of ellipsoids is not a new concept. In fact, if $\mathcal{E}_1, \mathcal{E}_2$ are ellipsoids then $g(\mathcal{E}_1,\mathcal{E}_2)$ is also an ellipsoid, and $u_{g(\mathcal{E}_1,\mathcal{E}_2)} = u_{\mathcal{E}_1} \# u_{\mathcal{E}_2}$ (this property was also proved in [14]). Note that this property is somewhat surprising: because we do not have a property like $u_{\mathcal{E}_1+\mathcal{E}_2} = u_{\mathcal{E}_1} + u_{\mathcal{E}_2}$, the iteration process for the matrices and the ellipsoids is different. We do not have in general $u_{A_m} = u_m$, and in fact A_m is almost never an ellipsoid. Still, in the limit one obtains an ellipsoid, which is exactly the one represented by the matrix $u_{\mathcal{E}_1} \# u_{\mathcal{E}_2}$.

Despite its many useful properties, the geometric mean g does not satisfy one important property:

Definition 5. We say that a map $\Phi : \mathcal{K}_s^n \times \mathcal{K}_s^n \to \mathcal{K}_s^n$ has the scaling property if for every $K,T \in \mathcal{K}_s^n$ and every $\alpha, \beta > 0$ we have

$$\Phi(\alpha K, \beta T) = \sqrt{\alpha\beta}\,\Phi(K,T).$$

This is definitely a natural property to expect from a geometric mean. A 2-dimensional example, constructed by Magazinov for an appendix of [14], shows that the scaling property is not satisfied by the geometric mean as constructed in Definition 3. In [18], the second author constructed a variant of the geometric mean which shares its good properties and also has the scaling property. The construction uses a new concept of geometric Banach limits for sequences of convex bodies:

Definition 6. Let $\mathcal{BK}^n$ denote the class of uniformly bounded sequences of convex bodies:

$$\mathcal{BK}^n = \left\{ \{K_m\}_{m=1}^{\infty} : \begin{array}{c} \text{There exists } r, R > 0 \text{ such that} \\ r \cdot B_2^n \subseteq K_m \subseteq R \cdot B_2^n \text{ for all } m \end{array} \right\}.$$

A geometric Banach limit is a function $L : \mathcal{BK}^n \to \mathcal{K}_s^n$ with the following properties:

1. L is shift invariant: $L(\{K_m\}) = L(\{K_{m+1}\})$.
2. If $K_m \to K$ in the Hausdorff metric then $L(\{K_m\}) = K$.
3. If $K_m \supseteq T_m$ for all m then $L(\{K_m\}) \supseteq L(\{T_m\})$.
4. For any invertible linear map u we have $L(\{uK_m\}) = uL(\{K_m\})$.
5. $L(\{\lambda K_m\}) = \lambda L(\{K_m\})$ for all $\lambda > 0$.
6. $L(\{K_n^\circ\}) = L(\{K_n\})^\circ$.

It is the last property that makes the construction of a geometric Banach limit a delicate matter. Surprisingly, the construction of L uses the geometric mean of convex bodies, even though they are not mentioned in the definition. For the full details, as well as the construction of the new geometric mean, the reader may consult [18].

We would like to now present another variant of g that has the scaling property, which may be simpler than the one constructed in [18]. However, for our construction it is important that the bodies are centrally symmetric, while the Banach limit construction works just as well for non-symmetric convex bodies. To present the construction we will need the following definition:

Definition 7. For $K, T \in \mathcal{K}_s^n$, the upper ellipsoidal envelope of K and T is

$$\overline{g}(K,T) = \bigcap \{g(\mathcal{E}_1, \mathcal{E}_2) : \ K \subseteq \mathcal{E}_1 \text{ and } T \subseteq \mathcal{E}_2\}.$$

Similarly, the lower ellipsoidal envelope of K and T is

$$\underline{g}(K,T) = \operatorname{conv} \bigcup \{g(\mathcal{E}_1, \mathcal{E}_2) : \ K \supseteq \mathcal{E}_1 \text{ and } T \supseteq \mathcal{E}_2\},$$

where conv denotes the convex hull.

Using the explicit formula (2.1) it is easy to check that the matrix geometric mean has the scaling property:

$$(\alpha u) \# (\beta v) = \sqrt{\alpha\beta} \cdot (u \# v).$$

The relation $u_{g(\mathcal{E}_1, \mathcal{E}_2)} = u_{\mathcal{E}_1} \# u_{\mathcal{E}_2}$ then implies that the geometric mean has the scaling property for ellipsoids. It follows that the ellipsoidal envelopes also have the scaling property.

However, each of the ellipsoidal envelopes by itself is not a good candidate to be a "geometric mean". For example, there is no reason for the relation $\overline{g}(K, K^\circ) = B_2^n$ to be true for an arbitrary $K \in \mathcal{K}_s^n$. This is a fundamental property which corresponds to the numerical fact that the geometric mean of x and $\frac{1}{x}$ is 1.

It turns out that even though each of the envelopes by itself is not a good geometric mean, they can be combined to create a very good candidate:

Definition 8. The ellipsoidal geometric mean of K and T is

$$G(K,T) = g\left(\overline{g}\,(K,T)\,,\underline{g}(K,T)\right).$$

As promised, the ellipsoidal geometric mean satisfies all of the basic properties of the original geometric mean, and also has the scaling property:

Theorem 9. *The ellipsoidal geometric mean has the following properties:*

1. $G(K,K) = K$.
2. *G is symmetric in its arguments:* $G(K,T) = G(T,K)$.
3. *G is monotone in its arguments: If* $K_1 \subseteq K_2$ *and* $T_1 \subseteq T_2$ *then* $G(K_1,T_1) \subseteq G(K_2,T_2)$.
4. *G satisfies the harmonic mean – geometric mean – arithmetic mean inequality*
$$\left(\frac{K^\circ + T^\circ}{2}\right)^\circ \subseteq G(K,T) \subseteq \frac{K+T}{2}.$$
5. $[G(K,T)]^\circ = G\left(K^\circ, T^\circ\right)$.
6. $G(K,K^\circ) = B_2^n$.
7. *For any linear map u we have* $G(uK,uT) = u\left(G(K,T)\right)$.
8. *G has the scaling property:* $G(\alpha K, \beta T) = \sqrt{\alpha\beta}\, G(K,T)$.

Proof. Properties (2), (3) and (7) are obvious from the corresponding properties of g. For (5) notice that

$$\overline{g}(K,T)^\circ = \underline{g}(K^\circ,T^\circ).$$

Hence

$$\begin{aligned}[G(K,T)]^\circ &= g\left(\overline{g}\,(K,T)\,,\underline{g}(K,T)\right)^\circ = g\left(\overline{g}\,(K,T)^\circ\,,\underline{g}(K,T)^\circ\right) \\ &= g\left(\overline{g}\,(K^\circ,T^\circ)\,,\underline{g}(K^\circ,T^\circ)\right) = G(K^\circ,T^\circ).\end{aligned}$$

Property (6) is a corollary of (5): We have $G(K,K^\circ)^\circ = G\left(K^\circ, K^{\circ\circ}\right) = G(K,K^\circ)$, and it is well known that the only solution to the equation $X = X^\circ$ is $X = B_2^n$.

To prove property (4), fix $\epsilon > 0$ and a unit vector $\theta \in \mathbb{R}^n$. Choose an ellipsoids $\mathcal{E}_1$ such that $\mathcal{E}_1 \supseteq K$ and $h_{\mathcal{E}_1}(\theta) \leq h_K(\theta) + \epsilon$ (To see that such an ellipsoid exists, take the "supporting slab" $\{x \in \mathbb{R}^n :\, |\langle x,\theta\rangle| \leq h_k(\theta)\}$ and approximate it by an ellipsoid). Similarly, choose an ellipsoid $\mathcal{E}_2$ such that $\mathcal{E}_2 \supseteq K$ and $h_{\mathcal{E}_2}(\theta) \leq h_K(\theta) + \epsilon$. It follows that

$$\begin{aligned}h_{\overline{g}(K,T)}(\theta) \leq h_{g(\mathcal{E}_1,\mathcal{E}_2)}(\theta) \leq h_{\frac{\mathcal{E}_1+\mathcal{E}_2}{2}}(\theta) &= \frac{h_{\mathcal{E}_1}(\theta) + h_{\mathcal{E}_2}(\theta)}{2} \\ &\leq \frac{h_K(\theta) + h_T(\theta)}{2} + \epsilon = h_{\frac{K+T}{2}}(\theta) + \epsilon.\end{aligned}$$

Since this is true for all $\epsilon > 0$ and all directions θ we conclude that $\overline{g}(K,T) \subseteq \frac{K+T}{2}$. Since the same is trivially true for $\underline{g}(K,T)$ we may conclude that

$$G(K,T) = g\left(\overline{g}\left(K,T\right), \underline{g}(K,T)\right) \subseteq g\left(\frac{K+T}{2}, \frac{K+T}{2}\right) = \frac{K+T}{2}.$$

Applying the same inequality to K° and T°we see that

$$G\left(K,T\right)^\circ = G\left(K^\circ, T^\circ\right) \subseteq \frac{K^\circ + T^\circ}{2},$$

and the harmonic mean – geometric mean inequality follows by taking the polar of both sides. This completes the proof of property (4). Of course, (1) follows immediately.

Finally, for property (8), we already explained why the ellipsoidal envelopes have the scaling property. But then

$$\begin{aligned} G(\alpha K, \beta T) &= g\left(\overline{g}\left(\alpha K, \beta T\right), \underline{g}(\alpha K, \beta T)\right) = g\left(\sqrt{\alpha\beta} \cdot \overline{g}\left(K,T\right), \sqrt{\alpha\beta} \cdot \underline{g}(K,T)\right) \\ &= \sqrt{\alpha\beta} \cdot g\left(\overline{g}\left(K,T\right), \underline{g}(K,T)\right) = \sqrt{\alpha\beta} \cdot G(K,T), \end{aligned}$$

and the proof is complete. $\square$

We do not know if a map $\Phi : \mathcal{K}_s^n \times \mathcal{K}_s^n \to \mathcal{K}_s^n$ satisfying properties (1)–(8) of the above theorem must coincide with G.

We conclude this section with several notes. First, as the unique self-polar convex body, the Euclidean ball B_2^n plays the role of the number 1 or the identity matrix. It follows that we may think of the body $G(K, B_2^n)$ as the square root $\sqrt{K}$. It is interesting to notice that even though $\sqrt{K}$ is defined for every $K \in \mathcal{K}_s^n$, the equation $\sqrt{X} = K$ does not always have a solution. This is essentially because

$$d_{BM}\left(\sqrt{X}, B_2^n\right) \leq \sqrt{d_{BM}\left(X, B_2^n\right)},$$

where d_{BM} denotes the Banach-Mazur distance (see [14]). Combined with John's theorem, we see that $d_{BM}\left(\sqrt{X}, B_2^n\right) \leq n^{\frac{1}{4}}$, so $\sqrt{X}$ can never be a cube for example. In other words, even though every convex body has a square root, not every convex body has a square.

There exist in the literature other attempts to define the geometric mean of two convex bodies K and T. In [5], Böröczky, Lutwak, Yang and Zhang construct the following "0-mean", or "logarithmic mean", of convex bodies:

$$L(K,T) = \left\{x \in \mathbb{R}^n : \ \langle x, \theta\rangle \leq \sqrt{h_K(\theta) h_T(\theta)} \text{ for all } \theta \in S^{n-1}\right\}$$

(their construction is for arbitrary weights λ and $1-\lambda$. Here we only cite the symmetric case $\lambda = \frac{1}{2}$). In other words, $L = L(K,T)$ is the largest convex function such that $h_L(\theta) \leq \sqrt{h_K(\theta) h_T(\theta)}$ for all $\theta \in S^{n-1}$.

This definition is similar in many ways to the upper ellipsoidal envelope $\overline{g}(K,T)$. To see this, let

$$\begin{aligned} S_1 &= \{x : \ |\langle x, \theta\rangle| \leq a\} \\ S_2 &= \{x : \ |\langle x, \eta\rangle| \leq b\} \end{aligned}$$

be two slabs. Even though $S_1, S_2 \notin \mathcal{K}_s^n$, since they are not compact, one may approximate them by ellipsoids and arrive at a natural definition for $g(S_1, S_2)$. The result will be that $g(S_1, S_2) = \mathbb{R}^n$ whenever $\theta \neq \eta$, and

$$g\left(S_1, S_2\right) = \left\{x : \ |\langle x, \theta\rangle| \leq \sqrt{ab}\right\}$$

if $\theta = \eta$. From here we see that

$$L(K, T) = \bigcap \left\{g(S_1, S_2) : \ K \subseteq S_1 \text{ and } T \subseteq S_2\right\},$$

which is very similar to the definition of $\overline{g}(K, T)$. It follows immediately that $\overline{g}(K, T) \subseteq L(K, T)$. Like $\overline{g}$ the mean L will also have the scaling property, but does not satisfy the polarity property $[g(K, T)]^\circ = g\left(K^\circ, T^\circ\right)$.

Another possible "geometric mean" was studied by Cordero-Erausquin and Klartag in [7], following a previous work of Semmes ([21]). Let $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}$ be (sufficiently smooth) convex functions. A p-interpolation between u_0 and u_1 is a function $u : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ such that $u(0, x) = u_0(x)$, $u(1, x) = u_1(x)$, and $u(t, x)$ satisfies the PDE

$$\partial^2_{tt} u = \frac{1}{p} \left\langle \left(\mathrm{Hess}_x u\right)^{-1} \nabla \partial_t u, \nabla \partial_t u \right\rangle .$$

Here we will care about the case $p = 2$. Given u_0 and u_1 it is not clear that this PDE has a solution, let alone a unique solution. However, it is not hard to check that if $u_0 = \frac{1}{2} h_K^2$ and $u_1 = \frac{1}{2} h_T^2$ for some bodies K and T, then $u_t = \frac{1}{2} h_{R_t}^2$ (assuming it exists) for some family of convex bodies $R_t = R_t\left(K, T\right)$. The body $R_{1/2}(K, T)$ is a possible candidate for the geometric mean whenever it is well-defined.

Finally, once the geometric mean is defined, one may use it for other constructions. For example, the Gauss arithmetic-geometric mean in the same way it is done for numbers (see, e.g. [16]): Given A_0, B_0 we set

$$A_{n+1} = \frac{A_n + B_n}{2} \qquad B_{n+1} = G(A_n, B_n)$$

The common limit $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n$, which always exists, is the arithmetic-geometric mean of A_0 and B_0. We will denote it by $M_{ag}(A_0, B_0)$. The geometric-harmonic mean, $M_{hg}(A_0, B_0)$, is defined similarly.

Special properties of these two constructions seem interesting to us. For example, it is proved in [16] that for numbers we have

$$M_{gh}(N, 1) = \frac{2}{\pi} \log 4N + O\left(1/N^2\right).$$

Is there an analogue for this result for $M_{gh}(K, B_2^n)$?

3 Powers of convex bodies

In the previous section we saw how it is possible to first compute the geometric mean for ellipsoids, and then use its good properties to construct a geometric mean for arbitrary convex bodies. One may use the same idea to define other functions on the class of convex bodies: because of the identification between ellipsoids and positive definite matrices, we can apply many standard functions on the class of ellipsoids.

We consider in this article the case of the power map $x \mapsto x^\alpha$. For ellipsoids, the definition is obvious:

Definition 10. If $\mathcal{E}$ is an ellipsoid and $\alpha \in \mathbb{R}$, we define the ellipsoid $\mathcal{E}^\alpha$ by the relation $u_{\mathcal{E}^\alpha} = (u_\mathcal{E})^\alpha$.

This definition makes sense for all α, because u^α is well-defined and positive definite for any positive definite matrix u. However, we will concentrate on the case $0 < \alpha < 1$, because in this case the power map is operator monotone:

Definition 11. A map f on the class of positive definite matrices is called operator monotone if $u \succcurlyeq v$ implies $f(u) \succcurlyeq f(v)$.

A proof of this fact that u^α is operator monotone whenever $0 < \alpha < 1$ can be found in [4], Theorem V.1.9. For $\alpha > 1$ the function $u \mapsto u^\alpha$ is not operator monotone. In particular, the square $u \mapsto u^2$ is not operator monotone – see example V.I.2 in [4]. This is related to our previous remark that not every convex body has a square.

From the monotonicity of $u \mapsto u^\alpha$ the following result is obvious:

Proposition 12. *The maps $\mathcal{E} \mapsto \mathcal{E}^\alpha$ defined on the class of ellipsoids have the following properties:*

1. *For every $0 < \alpha < 1$, if $\mathcal{E}_1 \subseteq \mathcal{E}_2$ then $\mathcal{E}_1^\alpha \subseteq \mathcal{E}_2^\alpha$.*
2. *For every $0 < \alpha < 1$, every ellipsoid $\mathcal{E}$ and every $\lambda > 0$ we have $(\lambda\mathcal{E})^\alpha = \lambda^\alpha \mathcal{E}^\alpha$.*
3. *For every $0 < \alpha, \beta < 1$ and every ellipsoid $\mathcal{E}$, $(\mathcal{E}^\alpha)^\beta = \mathcal{E}^{\alpha\beta}$.*

We would like to extend this power map to all centrally symmetric convex bodies. Thanks to the monotonicity of the power function, we may use the idea of ellipsoidal envelopes. We choose rather arbitrarily to work with upper envelopes and define:

Definition 13. For every $0 < \alpha < 1$ and every $K \in \mathcal{K}_s^n$ we set

$$P_\alpha(K) = \bigcap \{\mathcal{E}^\alpha : \ K \subseteq \mathcal{E}\}.$$

The map $K \mapsto P_\alpha(K)$ is obviously monotone. Furthermore, if $\mathcal{E}$ is any ellipsoid such that $\mathcal{E} \supseteq K$, then $\mathcal{E}^\alpha \supseteq P_\alpha(K)$, which implies that $\mathcal{E}^{\alpha\beta} = (\mathcal{E}^\alpha)^\beta \supseteq P_\beta(P_\alpha(K))$. Intersecting over all $\mathcal{E}$ we see that $P_{\alpha\beta}(K) \supseteq P_\beta(P_\alpha(K))$. Unfortunately, in general there is no reason for the equality $P_{\alpha\beta}(K) = P_\beta(P_\alpha(K))$ to hold.

To fix this problem, we refine our definition of the power. Our construction will be similar in spirit to the construction of the integral using Darboux sums. Fix some $0 < \alpha < 1$, and let Π be a finite partition of $[\alpha, 1]$:

$$\Pi : \ \alpha = t_0 < t_1 < \cdots < t_m = 1.$$

Setting $s_i = t_{i-1}/t_i$ for $i = 1, 2, ..m$, we define

$$P_\Pi(K) = (P_{s_1} \circ P_{s_2} \circ \cdots \circ P_{s_m})(K),$$

where $\circ$ denotes the composition.

We say that a partition Π is a refinement of $\widetilde{\Pi}$ if $\Pi \supseteq \widetilde{\Pi}$, i.e. Π is obtained from $\widetilde{\Pi}$ by adding points. As the partition Π becomes more refined, the body $P_\Pi(K)$ becomes smaller:

Lemma 14. *Assume $\Pi \supseteq \widetilde{\Pi}$ are partitions of $[\alpha, 1]$. Then for every convex body $K \in \mathcal{K}_s^n$ one has $P_\Pi(K) \subseteq P_{\widetilde{\Pi}}(K)$.*

Proof. Of course, it is enough to prove the result when $|\Pi| = \left|\widetilde{\Pi}\right| + 1$. Let

$$\Pi: \ \alpha = t_0 < t_1 < \cdots < t_m = 1$$

be a partition, and assume $\widetilde{\Pi}$ is obtained from Π by removing the point t_k. Denote by Π_1 the partition $\{t_0, t_1, \ldots, t_{k-1}\}$ and by Π_2 the partition $\{t_{k+1}, t_{k+2}, \ldots, t_m\}$. Then

$$P_\Pi = P_{\Pi_1} \circ P_{t_k/t_{k-1}} \circ P_{t_{k+1}/t_k} \circ P_{\Pi_2},$$

while

$$P_{\widetilde{\Pi}} = P_{\Pi_1} \circ P_{t_{k+1}/t_{k-1}} \circ P_{\Pi_2}$$

(the operators P_{Π_1} and P_{Π_2} are defined in the obvious way, even though Π_1 and Π_2 are not partitions of $[\alpha, 1]$).

As we already explained, for every convex body T we have

$$P_{t_k/t_{k-1}} \circ P_{t_{k+1}/t_k}(T) \subseteq P_{t_{k+1}/t_{k-1}}(T).$$

Choosing $T = P_{\Pi_2}(K)$, and using the fact that P_{Π_1} is monotone, we conclude the proof. $\square$

We may now define:

Definition 15. For every $K \in \mathcal{K}_s^n$ and $0 < \alpha < 1$ we define

$$K^\alpha = \bigcap_\Pi P_\Pi(K),$$

where the intersection is taken over all partitions of $[\alpha, 1]$.

Notice that if $\mathcal{E}$ is an ellipsoid then $P_\Pi(\mathcal{E}) = \mathcal{E}^\alpha$ for every partition Π of $[\alpha, 1]$, which means this definition is really an extension of Definition 10. Of course, we did not need Lemma 14 for K^α to be well-defined, as one can take intersections of arbitrary families of convex bodies. However, Lemma 14 is crucial for the following result, which shows that we can approximate K^α using sets of the form $P_\Pi(K)$:

Proposition 16. *Fix $K \in \mathcal{K}_s^n$ and $0 < \alpha < 1$. Then for every $\epsilon > 0$ one may find a partition Π of $[\alpha, 1]$ such that*

$$K^\alpha \subseteq P_\Pi(K) \subseteq (1 + \epsilon) K^\alpha.$$

Proof. The inequality $K^\alpha \subseteq P_\Pi(K)$ holds trivially for every partition Π, so we only need to prove the second inequality.

To prove it, assume by contradiction that $P_\Pi(K) \nsubseteq (1+\epsilon) K^\alpha$ for every partition Π. Set

$$A_\Pi = P_\Pi(K) \setminus \operatorname{int}((1+\epsilon) K^\alpha) \neq \emptyset,$$

where int denotes the interior of a set. We claim that the family $\{A_\Pi\}_\Pi$ has the finite intersection property: the intersection of finitely many sets $A_{\Pi_1}, A_{\Pi_2}, \ldots, A_{\Pi_m}$ is never empty. Indeed, if we denote $\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_m$ then $A_\Pi \neq \emptyset$ by the assumption, and $A_\Pi \subseteq A_{\Pi_i}$ for $i = 1, 2, .., m$ by Lemma 14.

Since the sets A_Π are all compact, the finite intersection property implies that $\bigcap_\Pi A_\Pi \neq \emptyset$. If we choose a point a in this intersection, then on the one hand $a \notin \operatorname{int}((1+\epsilon) K^\alpha)$, and on the other hand $a \in P_\Pi(K)$ for all Π, which implies that $a \in \bigcap_\Pi P_\Pi(K) = K^\alpha$. Since $K^\alpha \subset \operatorname{int}((1+\epsilon) K^\alpha)$ we arrived at a contradiction, and the proof is complete. $\square$

The main result of this section is that the power map $K \mapsto K^\alpha$, as defined in Definition 15, has the following properties (as it had for ellipsoids):

Theorem 17. *The maps $K \mapsto K^\alpha$ defined on $\mathcal{K}_s^n$ have the following properties:*

1. *For every $0 < \alpha < 1$, if $K \subseteq T$ then $K^\alpha \subseteq T^\alpha$.*
2. *For every $0 < \alpha < 1$, every $K \in \mathcal{K}_s^n$ and every $\lambda > 0$ we have $(\lambda K)^\alpha = \lambda^\alpha K^\alpha$.*
3. *For every $0 < \alpha, \beta < 1$ and every $K \in \mathcal{K}_s^n$ we have $(K^\alpha)^\beta = K^{\alpha\beta}$.*

Proof. Properties (1) and (2) are obvious: these properties pass from $P_\alpha(K)$ to $P_\Pi(K)$ and then to K^α.

Next we prove property (3). Let

$$\Pi: \ \alpha\beta = t_0 < t_1 < \cdots < t_m = 1$$

be any partition of $[\alpha\beta, 1]$, and let k be the maximal index such that $t_k < \alpha$. Let $\Pi_1 = \{t_0, t_1, \ldots, t_k, \alpha\}$ be a partition of $[\alpha\beta, \alpha]$, and $\Pi_2 = \{\alpha, t_{k+1}, t_{k+2}, \ldots, t_m\}$ be a partition of $[\alpha, 1]$. Finally, let

$$\widetilde{\Pi}_1 = \left\{\frac{t_0}{\alpha}, \frac{t_1}{\alpha}, \ldots, \frac{t_k}{\alpha}, 1\right\}$$

be a partition of $[\beta, 1]$. Notice that by definition we have $P_{\Pi_1} = P_{\widetilde{\Pi}_1}$. Since $\Pi_1 \cup \Pi_2 \supseteq \Pi$ we have

$$P_\Pi(K) \supseteq P_{\Pi_1 \cup \Pi_2}(K) = P_{\Pi_1}(P_{\Pi_2}(K)) = P_{\widetilde{\Pi}_1}(P_{\Pi_2}(K)) \supseteq P_{\widetilde{\Pi}_1}(K^\alpha) \supseteq (K^\alpha)^\beta.$$

Since this is true for every partition Π of $[\alpha\beta, 1]$, we make intersect over all such partitions and conclude that $K^{\alpha\beta} \supseteq (K^\alpha)^\beta$.

For the proof of the opposite inequality we need to use Proposition 16. Fix some $\epsilon > 0$. There exists a partition Π_2 of $[\alpha, 1]$ such that $P_{\Pi_2}(K) \subseteq (1+\epsilon) K^\alpha$. Similarly, there exists a partition Π_1 of $[\beta, 1]$ such that

$$P_{\Pi_1}(P_{\Pi_2}(K)) \subseteq (P_{\Pi_2}(K))^\beta \subseteq ((1+\epsilon) K^\alpha)^\beta = (1+\epsilon)^\beta \cdot (K^\alpha)^\beta.$$

On the other hand, $\Pi = \alpha\Pi_1 \cup \Pi_2$ is a partition of $[\alpha\beta, 1]$, so

$$P_{\Pi_1}\left(P_{\Pi_2}(K)\right) = P_{\alpha\Pi_1}\left(P_{\Pi_2}(K)\right) = P_{\Pi}(K) \supseteq K^{\alpha\beta}.$$

Combining the last two inclusions we see that $K^{\alpha\beta} \subseteq (1+\epsilon)^{\beta}(K^{\alpha})^{\beta}$. Since this is true for every $\epsilon > 0$ we have $K^{\alpha\beta} \subseteq (K^{\alpha})^{\beta}$ and the proof is complete. $\square$

The same method can be used to define $f(K)$ for other operator monotone functions f. As an important example, the function $u \mapsto \log u$ is operator monotone, which means that one can define $\log K$ using an ellipsoidal envelope. Notice however that the matrix $\log u$ is positive definite if and only if $u \succcurlyeq Id$, where Id is the identity map. It follows that we may define $f(\mathcal{E})$ only for ellipsoids $\mathcal{E}$ such that $\mathcal{E} \supseteq B_2^n$. Hence the natural domain of the logarithm is all convex bodies $K \in \mathcal{K}_s^n$ such that $K \supseteq B_2^n$. Unfortunately this definition of a logarithm does not seem to interact well with powers, in the sense that usually we do not have

$$\log\left(K^{\alpha}\right) = \alpha \log K$$

like we have for ellipsoids. We think that some modifications to the definitions may fix this problem.

In conclusion, we illustrated in this article just the very first steps in the development of an "irrational" theory of convexity. A lot of novel questions appear naturally in every step of this study. In [14] we explicitly formulated some of them. We also did not discuss in this note any problems involving the interplay between these new constructions and geometric (say, volume) inequalities.

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