A LETTER: THE LOG-BRUNN-MINKOWSKI INEQUALITY FOR COMPLEX BODIES

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We will use the following terminology: A real body $K \subseteq \mathbb{R}^n$ is the unit ball of a norm $\|\cdot\|$ on $\mathbb{R}^n$, i.e. a convex, origin symmetric, compact set with non-empty interior. Similarly, a complex body $K \subseteq \mathbb{C}^n$ is the unit ball of a norm $\|\cdot\|$ on $\mathbb{C}^n$. By identifying $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ we see that every complex body is also a real body, but not vice versa. In fact, a complex body $K \subseteq \mathbb{C}^n$ is a real body which is also symmetric with respect to complex rotations, i.e. if $z \in K$ implies that $e^{i\theta}z \in K$ for all $\theta \in \mathbb{R}$.

For a real body K, the support function of K is defined as $h_K(\theta) = \|\theta\|_K^* = \sup_{x \in K} \langle x, \theta \rangle$. Given two such bodies K and T and a number $0 \le \lambda \le 1$, we define the logarithmic mean of K and T by

$$L_\lambda(K,T) = \left\{ x \in \mathbb{R}^n : \ \langle x, \theta \rangle \le h_K(\theta)^{1-\lambda} h_T(\theta)^\lambda \text{ for all } \theta \in \mathbb{R}^n \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product.

The log-Brunn-Minkowski inequality states that $|L_\lambda(K,T)| \ge |K|^{1-\lambda} |T|^\lambda$, where $|\cdot|$ denotes the (Lebesgue) volume. It was conjectured by Böröczky, Lutwak, Yang and Zhang ([2]), who proved it for $K, T \subseteq \mathbb{R}^2$. Saroglou proved ([6]) that the inequality holds when K and T are n-dimensional real bodies which are unconditional with respect to the same basis.

The goal of this note is to explain why the log-Brunn-Minkowski inequality holds for complex bodies:

Theorem 1. *For complex bodies $K, T \subseteq \mathbb{C}^n$ and $0 \le \lambda \le 1$ we have $|L_\lambda(K,T)| \ge |K|^{1-\lambda} |T|^\lambda$.*

Theorem 1 will follow from a result of Cordero-Erausquin ([3]). In his work, Cordero-Erausquin proved a generalization of the Blaschke-Santaló inequality in the complex case. Specifically, he proved that for complex bodies $K, T \subseteq \mathbb{C}^n$ we have

$$(*) \qquad |K \cap T| \, |K^\circ \cap T| \le \left| B_2^{2n} \cap T \right|,$$

where K° is the polar body to K and $B_2^{2n} \subseteq \mathbb{C}^n$ is the unit Euclidean ball. As a side note we remark that proving the same inequality for general real bodies is an open problem – see [4] for a partial result and a short discussion.

Cordero-Erausquin's proved the inequality $(*)$ as a corollary of a general theorem about complex interpolation - see Theorem 3 below. The main point of this letter is the observation that the same general theorem also implies Theorem 1. This was apparently known to Cordero-Erausquin himself, but not to other researchers in the community who haven't studied the complex case carefully. Theorem 1 may be a strong indication that the log-Brunn-Minkowski conjecture is true in general. Alternatively, it may indicate the existence of a rich theory of geometric inequalities in the complex case.

Let us briefly recall the definition of complex interpolation. We will give the construction for the finite-dimensional case, following the presentation of [3], and refer the reader to [1] for a more detailed account. Set $S = \{z \in \mathbb{C} : \ 0 < \operatorname{Re} z < 1\}$, and define

$$\mathcal{F} = \left\{ f : \overline{S} \to \mathbb{C}^n : \ \begin{array}{l} f \text{ is bounded and continuous on } \overline{S} \text{ and analytic on } S \\ \text{such that } \lim\limits_{t \to \pm\infty} f(it) = \lim\limits_{t \to \pm\infty} f(1+it) = 0 \end{array} \right\}.$$

Given two norms $\|\cdot\|_0$ and $\|\cdot\|_1$ on $\mathbb{C}^n$, we define a norm on $\mathcal{F}$ by

$$\|f\|_{\mathcal{F}} = \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_0 , \sup_{t \in \mathbb{R}} \|f(1+it)\|_1 \right\}.$$

Finally, for $\lambda \in [0,1]$, we define the interpolated norm $\|\cdot\|_\lambda$ by

$$\|x\|_\lambda = \inf\left\{\|f\|_{\mathcal{F}} : \ f \in \mathcal{F}, \ f(\lambda) = x\right\}.$$

It is not hard to see that for $\lambda = 0, 1$ we recover the original norms $\|\cdot\|_0, \|\cdot\|_1$. The only other result we will need from the standard theory of complex interpolation is the following:

Proposition 2. *Let* $\|\cdot\|_0$, $\|\cdot\|_1$ *be norms on* $\mathbb{C}^n$ *and let* $\|\cdot\|_\lambda$ *be the interpolated norms. Then*

$$\|z\|_\lambda^* \le \left(\|z\|_0^*\right)^{1-\lambda} \left(\|z\|_1^*\right)^\lambda$$

for every $z \in \mathbb{C}^n$.

This inequality, with its simple proof, may be found for example in [5] as equation $(7.26)^*$.

If K is the unit ball of $\|\cdot\|_0$ and T is the unit ball of $\|\cdot\|_1$ we will write $C_\lambda(K,T)$ for the unit ball of $\|\cdot\|_\lambda$. Proposition 2 implies that $h_{C_\lambda(K,T)}(z) \le h_K(z)^{1-\lambda} h_T(z)^\lambda$ for all $z \in \mathbb{C}^n$, and hence $C_\lambda(K,T) \subseteq L_\lambda(K,T)$.

The main theorem of [3] is the following:

Theorem 3. *The function* $\lambda \longmapsto |C_\lambda(K,T)|$ *is log-concave on* $[0,1]$.

It is now easy to deduce Theorem 1, as

$$|L_\lambda(K,T)| \ge |C_\lambda(K,T)| \ge |C_0(K,T)|^{1-\lambda} \cdot |C_1(K,T)|^\lambda = |K|^{1-\lambda} |T|^\lambda.$$

References

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