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BANACH LIMIT IN CONVEXITY AND GEOMETRIC MEANS FOR CONVEX BODIES

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Abstract. In this note we construct Banach limits on the class of sequences of convex bodies. Surprisingly, the construction uses the recently introduced geometric mean of convex bodies. In the opposite direction, we explain how Banach limits can be used to construct a new variant of the geometric mean that has some desirable properties.

1. Introduction: Geometric means of convex bodies

We begin by fixing basic notation (for more background in convexity the reader may consult [9]). We denote by $\mathcal{K}_{(0)}^n$ the class of compact convex sets $K \subseteq \mathbb{R}^n$ such that 0 belongs to the interior of K. For $K \in \mathcal{K}_{(0)}^n$ its support function $h_K : \mathbb{R}^n \to (0, \infty)$ is defined by $h_K(\theta) = \sup_{x \in K} \langle x, \theta \rangle$. The set $\mathcal{K}_{(0)}^n$ has a natural structure of a cone, with addition being the Minkowski addition

$$K + T = \{x + y : \ x \in K, \ y \in T\},$$

and multiplication by scalar $\lambda > 0$ being given by the dilation

$$\lambda K = \{\lambda x : \ x \in K\}.$$

We put on $\mathcal{K}_{(0)}^n$ the Hausdorff metric, defined by

$$\begin{aligned} d_H(K, T) &= \min\{t > 0 : \ K \subseteq T + tB_2^n \text{ and } T \subseteq K + tB_2^n\} \\ &= \max_{\theta \in S^{n-1}} |h_K(\theta) - h_T(\theta)|. \end{aligned}$$

Here B_2^n denotes the unit Euclidean ball, and S^{n-1} denotes its boundary, the unit sphere. Whenever we talk about continuity or convergence in $\mathcal{K}_{(0)}^n$, we will implicitly equip $\mathcal{K}_{(0)}^n$ with this metric. The metric space $\mathcal{K}_{(0)}^n$ is "almost" complete – every Cauchy sequence $\{K_m\} \subseteq \mathcal{K}_{(0)}^n$ converges to a compact convex set K, but we may have $K \notin \mathcal{K}_{(0)}^n$ if 0 is not in the interior of K.

For $K \in \mathcal{K}_{(0)}^n$ the polar body $K^\circ \in \mathcal{K}_{(0)}^n$ is defined by

$$K^\circ = \{x \in \mathbb{R}^n : \ \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

The polarity map $K \mapsto K^\circ$ is a "duality" on $\mathcal{K}_{(0)}^n$, in the following formal sense:

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- It is order reversing: If $K \subseteq T$ then $K^\circ \supseteq T^\circ$.
- It is an involution: $(K^\circ)^\circ = K$.

In fact, the polarity map is essentially the only duality on $\mathcal{K}^n_{(0)}$, as we see from the following theorem:

Theorem 1. *Assume $\mathcal{T} : \mathcal{K}^n_{(0)} \to \mathcal{K}^n_{(0)}$ is an order reversing involution. Then there exists an invertible symmetric linear transformation $B \in GL(n)$ such that $\mathcal{T}K = BK^\circ$ for all $K \in \mathcal{K}^n_{(0)}$.*

On the class $\mathcal{K}^n_{(0)}$, this theorem was proved by Böröczky and Schneider [2]. Similar results on different classes of convex bodies were proved by Artstein-Avidan and Milman [1], and can be deduced from the work of Gruber [3].

The structure of an ordered cone with a duality transform appears often in mathematics, including in less geometric settings. The simplest example is the positive real numbers $\mathbb{R}_+$ themselves, with the usual addition, multiplication and order, and with the inversion $x \mapsto \frac{1}{x}$ as a duality. Another algebraic example is the class $\mathcal{M}^n_+$ of $n \times n$ positive-definite matrices. Here the addition and multiplication by scalar are the obvious choices, and the order is the matrix order $\preceq$, that is, $M_1 \preceq M_2$ if $M_2 - M_1$ is positive definite. The duality is the matrix inversion $M \mapsto M^{-1}$.

It turns out that there are surprising similarities between the algebraic classes of numbers and matrices and the geometric class of convex bodies. Thinking of the polar body K° as the "inverse" K^{-1} is often a good way to conjecture new results in convexity. Of course, once conjectured, these results should still be proved. In this note we will focus on one manifestation of this phenomenon: the geometric mean of convex bodies. We refer the reader to [7] and [8] for more interesting examples.

To explain the construction of the geometric mean of convex bodies, we first consider real numbers. For every $x, y > 0$ we define

$$\begin{aligned} a_0 &= x & h_0 &= y \\ a_{n+1} &= \frac{a_n + h_n}{2} & h_{n+1} &= \left(\frac{a_n^{-1} + h_n^{-1}}{2}\right)^{-1}. \end{aligned}$$

It is easy to prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} h_n = \sqrt{xy}$. A similar construction is known for positive definite matrices (see, e.g., [4] for a survey on the geometric mean of matrices).

Since the process above only uses addition and inversion, we may repeat it on the class of convex bodies. For $K, T \in \mathcal{K}^n_{(0)}$ we define

$$\begin{aligned} A_0 &= K & H_0 &= T \\ A_{n+1} &= \frac{A_n + H_n}{2} & H_{n+1} &= \left(\frac{A_n^\circ + H_n^\circ}{2}\right)^\circ. \end{aligned} \tag{1.1}$$

The following simple theorem is taken from [6]:

Theorem 2. *Fix $K, T \in \mathcal{K}^n_{(0)}$ and define sequences $\{A_n\}$ and $\{H_n\}$ according to (1.1). Then $\{A_n\}$ is decreasing, $\{H_n\}$ is increasing, and the limits $\lim_{n\to\infty} A_n$ and $\lim_{n\to\infty} H_n$ exist and are equal.*

The joint limit from the previous theorem is called the *geometric mean* of K and T and is denoted by $g(K, T)$. Let us list some properties of the geometric mean:

Proposition 3.

(1) $g(K, K) = K$.
(2) *g is symmetric in its arguments:* $g(K, T) = g(T, K)$.
(3) *g is monotone in its arguments:*
if $K_1 \subseteq K_2$ *and* $T_1 \subseteq T_2$ *then* $g(K_1, T_1) \subseteq g(K_2, T_2)$.
(4) *g is continuous in its arguments.*
(5) *g satisfies the harmonic–geometric–arithmetic mean inequality*

$$\left(\frac{K^\circ + T^\circ}{2}\right)^\circ \subseteq g(K, T) \subseteq \frac{K + T}{2}.$$

(6) $[g(K, T)]^\circ = g(K^\circ, T^\circ)$.
(7) $g(K, K^\circ) = B_2^n$.
(8) *For any linear map u we have* $g(uK, uT) = u(g(K, T))$.
In particular $g(\lambda K, \lambda T) = \lambda g(K, T)$.

All of the above properties were proved in [6], with the exception the continuity property (4). We will now prove that this property follows from the others:

Proof. Assume $K_m \to K$ and $T_m \to T$ for some $K, T \in \mathcal{K}^n_{(0)}$. It follows that there exist $r, R > 0$ such that $rB_2^n \subseteq K_m, T_m \subseteq RB_2^n$ for all m, and then we also have $rB_2^n \subseteq K, T \subseteq RB_2^n$. By properties (3) and (1) of the geometric mean we see that $rB_2^n \subseteq g(K, T) \subseteq RB_2^n$ and $rB_2^n \subseteq g(K_m, T_m) \subseteq RB_2^n$ for all m.

For a fixed m, let d denote the maximum of $d_H(K_m, K)$ and $d_H(T_m, T)$. It follows that

$$K_m \subseteq K + d \cdot B_2^n \subseteq K + \frac{d}{r} K = \left(1 + \frac{d}{r}\right) K,$$

and similarly $T_m \subseteq \left(1 + \frac{d}{r}\right) T$. Hence

$$g(K_m, T_m) \subseteq g\left(\left(1 + \tfrac{d}{r}\right) K, \left(1 + \tfrac{d}{r}\right) T\right) = \left(1 + \tfrac{d}{r}\right) g(K, T) \subseteq g(K, T) + d\tfrac{R}{r} \cdot B_2^n,$$

where we used properties (3) and (8) of the geometric mean. Repeating the same argument with the roles of K_m, T_m and K, T reversed, we conclude that

$$d_H\left(g(K_m, T_m), g(K, T)\right) \leq \frac{R}{r} \cdot \max\left\{d_H(K_m, K), d_H(T_m, T)\right\}.$$

Letting $m \to \infty$ we see that $g(K_m, T_m) \to g(K, T)$, so g is continuous. $\square$

One natural property we would expect the geometric mean to satisfy is the scaling property $g(\alpha K, \beta T) = \sqrt{\alpha\beta} g(K, T)$. Unfortunately, it turns out that in general this is simply false. A 2-dimensional counterexample was computed by Alexander Magazinov and appears in the appendix of [6]. In fact, in Magazinov's example the body $g(\alpha K, \beta T)$ is not even homothetic to $g(K, T)$.

In section 3 we will construct a modification of g that satisfies all the properties from Proposition 3 and has the scaling property. Before we do so however, we will develop in Section 2 the notion of Banach limits for sequences of convex bodies. As we will see, there is a surprising relation between geometric means and Banach limits.

2. Banach limits

In this section we will discuss the construction of a Banach limit for sequences of convex bodies. At first this section may appear to be completely unrelated to the previous section, but the connection with the geometric mean will soon become apparent.

Let us begin by recalling the classical definition of a Banach limit. As usual, we denote by ℓ^∞ the space of all bounded sequences of real numbers. Informally, a Banach limit is a way to assign a "limit" to every sequence $\{a_m\} \in \ell_\infty$, in a way that is linear and shift invariant. More formally we have the following theorem:

Theorem 4. *There exists a* linear *functional* $LIM : \ell_\infty \to \mathbb{R}$ *such that*

(1) *LIM is shift invariant:* $LIM(\{a_m\}) = LIM(\{a_{m+1}\})$.
(2) *For every* $\{a_m\}$ *we have* $\liminf_{m\to\infty} a_m \le LIM(\{a_m\}) \le \limsup_{m\to\infty} a_m$. *In particular, for convergent sequences LIM agrees with the standard limit.*
(3) *If* $a_m \ge b_m$ *for all* m *then then* $LIM(\{a_m\}) \ge LIM(\{b_m\})$.

For a proof of this theorem see, e.g., Section 4.2 of [5]. The functional *LIM* is called a Banach limit on ℓ^∞. Since the proof of the theorem uses the Hahn-Banach theorem, *LIM* is highly non-constructive and non-unique.

Now we turn our attention to convex bodies. Let us denote by $\mathcal{BK}^n$ the space of uniformly bounded sequences of convex bodies. More explicitly,

$$\mathcal{BK}^n = \left\{ \{K_m\}_{m=1}^\infty \ : \ \begin{array}{c} \text{There exists } r, R > 0 \text{ such that} \\ r \cdot B_2^n \subseteq K_m \subseteq R \cdot B_2^n \text{ for all } m \end{array} \right\}.$$

For the next theorem the existence of the lower bound $r \cdot B_2^n$ is not important; only the existence of the upper bound $R \cdot B_2^n$ is. However, the uniform lower bound will be crucial later.

The following theorem ensures the existence of a certain functional, which we call a *linear Banach limit* on convex bodies.

Theorem 5. *There exists a map* $L : \mathcal{BK}^n \to \mathcal{K}^n_{(0)}$ *with the following properties:*

(1) *L is shift invariant:* $L(\{K_m\}) = L(\{K_{m+1}\})$.
(2) *If* $K_m \to K$ *in the Hausdorff metric then* $L(\{K_m\}) = K$.
(3) *If* $K_m \supseteq T_m$ *for all* m *then* $L(\{K_m\}) \supseteq L(\{T_m\})$.
(4) *For any invertible linear map* u *we have* $L(\{uK_m\}) = uL(\{K_m\})$.
(5) $L(\{\lambda K_m\}) = \lambda L(\{K_m\})$ *for all* $\lambda > 0$.
(6) $L(\{K_m + T_m\}) = L(\{K_m\}) + L(\{T_m\})$.

Of course, the reason we call L linear is because it satisfies properties (5) and (6). Like the situation for sequences of numbers, L is highly non-unique.

Proof. We fix a standard Banach limit *LIM* on ℓ_∞. For every sequence $\{K_n\} \subseteq \mathcal{BK}^n$ we define $K = LIM(\{K_n\})$ implicitly by the relation

$$h_K(\theta) = LIM(\{h_{K_n}(\theta)\}).$$

First we need to check that this indeed defines a convex body: we have

$$\begin{aligned} h_K(\alpha\theta + \beta\eta) &= LIM(\{h_{K_n}(\alpha\theta + \beta\eta)\}) \\ &\le LIM(\{\alpha h_{K_n}(\theta) + \beta h_{K_n}(\eta)\}) \\ &= \alpha LIM(\{h_{K_n}(\theta)\}) + \beta LIM(\{h_{K_n}(\eta)\}) = \alpha h_K(\theta) + \beta h_K(\theta), \end{aligned}$$

where we used both the monotonicity and the linearity of *LIM*. Hence the body K is indeed well defined.

Properties (1), (3), (5), and (6) are obvious from the corresponding properties of *LIM*. For (2) we also need to remember that if $K_n \to K$ in the Hausdorff distance then in particular $h_{K_n}(\theta) \to h_K(\theta)$ for all θ. Finally, (4) is easily proved if we recall that $h_{uK_n}(\theta) = h_{K_n}(u^t\theta)$ and that the support function defines the convex body uniquely. □

For applications in convexity, and in particular for the application we have in mind, we would like our Banach limit to respect the notion of polarity. In other words, we would like to have $L(\{K_m^\circ\}) = L(\{K_m\})^\circ$ for any sequence $\{K_m\} \in \mathcal{BK}^n$. Unfortunately, it turns out that polarity was not mentioned in Theorem 5 for a good reason – one cannot add this assumption to the theorem.

To see that this is the case, assume that $L : \mathcal{BK}^n \to \mathcal{K}^n_{(0)}$ satisfies properties (1), (2) and (6). For any $A, B \in \mathcal{K}^n_{(0)}$ let us denote

$$K = L(\{A, B, A, B, A, B, \ldots\}).$$

By (1) we also have $K = L(\{B, A, B, A, B, A, \ldots\})$, and then by (6) and (2) we have

$$2K = L(\{A + B, B + A, A + B, B + A, \ldots\}) = A + B,$$

so $K = \frac{A+B}{2}$. In particular, if we take $B = A^\circ$ then

$$L(\{A^\circ, A, A^\circ, A, A^\circ, A \ldots\}) = L(\{A, A^\circ, A, A^\circ, A, A^\circ \ldots\}) = \frac{A + A^\circ}{2}.$$

Since for $A \neq B_2^n$ we have $\left(\frac{A+A^\circ}{2}\right)^\circ \neq \frac{A+A^\circ}{2}$, we see that the property $L(\{K_m^\circ\}) = L(\{K_m\})^\circ$ is not satisfied.

So, if we want our Banach limits to respect polarity, we must give up on one of their other properties. Properties (1) and (2) are crucial for our intuition for what a Banach limit is, and we do not want to remove them. Perhaps surprisingly, however, it turns out the additivity property (6) is not so important for us. Removing it, we may arrive at the following theorem:

Theorem 6. *There exists a map $\widetilde{L} : \mathcal{BK}^n \to \mathcal{K}^n_{(0)}$ satisfying properties (1)-(5) of Theorem 5 together with*

(6') $\widetilde{L}(\{K_m^\circ\}) = \widetilde{L}(\{K_m\})^\circ$.

We call $\widetilde{L}$ a geometric Banach limit *on $\mathcal{BK}^n$.*

Proof. Fix a linear Banach limit L on $\mathcal{BK}^n$, and define

$$\widetilde{L}(\{K_m\}) = g\left(L(\{K_m\}), L(\{K_m^\circ\})^\circ\right),$$

where g is the geometric mean.

The fact that (1) remains true is obvious, and the continuity property (2) holds because L, g, and the polarity map are all continuous. Similarly, (3) holds because g is monotone, (4) holds because g is linear equivariant, and (5) is a special case of (4).

Finally, for (6'), we have

$$\begin{aligned}\widetilde{L}\left(\{K_m^\circ\}\right) &= g\left(L\left(\{K_m^\circ\}\right), L\left(\{K_m^{\circ\circ}\}\right)^\circ\right) = g\left(L\left(\{K_m^{\circ\circ}\}\right)^\circ, L\left(\{K_m^\circ\}\right)\right) \\ &= g\left(L\left(\{K_m\}\right)^\circ, L\left(\{K_m^\circ\}\right)^{\circ\circ}\right) = g\left(L\left(\{K_m\}\right), L\left(\{K_m^\circ\}\right)^\circ\right)^\circ \\ &= \widetilde{L}\left(\{K_m\}\right)^\circ. \qquad \square\end{aligned}$$

Notice the surprising use of the geometric mean g in a theorem that did not mention means anywhere in its formulation.

Remark 7. As an example of the difference between the linear and the geometric Banach limit, fix two convex bodies $A, B \in \mathcal{K}_{(0)}^n$ and consider the periodic sequence $\{A, B, A, B, A, B, \ldots\}$. For the linear Banach limit we already saw that

$$L\left(\{A, B, A, B, \ldots\}\right) = \frac{A+B}{2}.$$

For the geometric Banach limit constructed in Theorem 6, however, the same sequence will converge to the geometric mean:

$$\widetilde{L}\left(\{A, B, A, B, \ldots\}\right) = g\left(\frac{A+B}{2}, \left(\frac{A^\circ + B^\circ}{2}\right)^\circ\right) = g(A, B).$$

Of course, one may also construct a Banach limit for which the sequence $\{A, B, A, B, A, B, \ldots\}$ converges to the harmonic mean of A and B (but will not satisfy the polarity property and will not be linear with respect to the Minkowski addition).

In the next section we will need to use Banach limits not just for sequences but for functions as well. The following variant of Theorem 4 appears in [5] as an exercise:

Theorem 8. *Denote by $B(\mathbb{R})$ the space of bounded functions $f : \mathbb{R} \to \mathbb{R}$. There exists a* linear *functional $LIM : B(\mathbb{R}) \to \mathbb{R}$ such that*

1. *LIM is shift invariant: If for $f \in B(\mathbb{R})$ and $v \in \mathbb{R}$ we define $f_v(t) = f(t+v)$, then $LIM(f) = LIM(f_v)$.*
2. *For every f we have $\liminf_{t\to\infty} f(t) \le LIM(f) \le \limsup_{t\to\infty} f(t)$.*
3. *If $f(t) \ge g(t)$ for all $x \in \mathbb{R}$ then $LIM(f) \ge LIM(g)$.*

Notice that our Banach limit is "at $+\infty$," meaning it agrees with $\lim_{t\to\infty} f(t)$ whenever this limit exists. One may similarly talk about Banach limits "at $-\infty$," or even at a finite point $t_0 \in \mathbb{R}$. Since we will not use such functionals, we use the convention that "Banach limit" always means "Banach limit at $+\infty$."

By using Theorem 8 instead of Theorem 4, one can also build Banach limits for body-valued functions. To be more exact, let us write

$$\mathcal{F}^n = \left\{ f : \mathbb{R} \to \mathcal{K}_{(0)}^n \ : \ \begin{array}{c}\text{There exists } r, R > 0 \text{ such that} \\ r \cdot B_2^n \subseteq f(t) \subseteq R \cdot B_2^n \text{ for all } t\end{array} \right\}.$$

Then one may construct a linear Banach limit $L : \mathcal{F}^n \to \mathcal{K}_{(0)}^n$ satisfying the obvious analogues of properties (1)-(6) from Theorem 5. Similarly, one may construct a geometric Banach limit $L : \mathcal{F}^n \to \mathcal{K}_{(0)}^n$ satisfying the properties (1)-(5) and (6').

3. A modified geometric mean

In the previous section we used the geometric mean of convex bodies in order to build the new notion of a geometric Banach limit. We will now do the opposite and use the newly constructed geometric Banach limit in order to build a new variant of the geometric mean. This new construction will share all the nice properties of the original geometric mean, but it will also have the scaling property. Here is the relevant definition:

Definition 9. Fix a geometric Banach limit $L : \mathcal{F}^n \to \mathcal{K}^n_{(0)}$. Given $K, T \in \mathcal{K}^n_{(0)}$, we define a function $f_{K,T} \in \mathcal{F}^n$ by

$$f_{K,T}(t) = g\big(e^t K, e^{-t} T\big).$$

We now define the Banach geometric mean $G(K,T)$ by $G(K,T) = g\big(Lf_{K,T}, Lf_{T,K}\big)$.

In order to show that the Banach geometric mean is well-defined, we need to prove that indeed $f_{K,T} \in \mathcal{F}^n$ for all $K, T \in \mathcal{K}^n_{(0)}$. To show this, fix $r, R > 0$ such that $rB_2^n \subseteq K, T \subseteq RB_2^n$ and notice that

$$f_{K,T}(t) = g\big(e^t K, e^{-t} T\big) \subseteq g\big(e^t RB_2^n, e^{-t} RB_2^n\big) = R \cdot g\big(e^t B_2^n, (e^t B_2^n)^\circ\big) = R \cdot B_2^n.$$

Similarly $f_{K,T}(t) \supseteq rB_2^n$, so $f_{K,T} \in \mathcal{F}^n$ and G is well-defined.

It appears as though the construction of $G(K,T)$ depends crucially on the choice of a geometric Banach limit. We will soon see that this is not the case, and in fact one may define $G(K,T)$ in a more constructive way. Let us begin, however, by stating the main properties of this construction:

Theorem 10. *The body $G(K,T)$ satisfies the following properties:*

1. $G(K,K) = K$.
2. *G is symmetric in its arguments: $G(K,T) = G(T,K)$.*
3. *G is monotone in its arguments:* *if $K_1 \subseteq K_2$ and $T_1 \subseteq T_2$ then $G(K_1,T_1) \subseteq G(K_2,T_2)$.*
4. *G is continuous in its arguments.*
5. *G satisfies the harmonic–geometric–arithmetic mean inequality*
$$\left(\frac{K^\circ + T^\circ}{2}\right)^\circ \subseteq G(K,T) \subseteq \frac{K+T}{2}.$$
6. $[G(K,T)]^\circ = G(K^\circ, T^\circ)$.
7. $G(K,K^\circ) = B_2^n$.
8. *For any linear map u we have $G(uK,uT) = u\,(G(K,T))$.*
9. *G has the scaling property: $G(\alpha K, \beta T) = \sqrt{\alpha\beta}\, G(K,T)$.*

Proof. (1), (2), (3), (4), (6), and (8) are immediate from the suitable properties of g and the geometric Banach limit L. (7) is an immediate corollary of (6).

To prove (5), we recall from [6] that if $C = g(A,B)$ then $h_C(\theta) \le \sqrt{h_A(\theta) h_B(\theta)}$ for every direction $\theta \in S^{n-1}$. It follows that

$$h_{f_{K,T}(t)}(\theta) \le \sqrt{e^t h_K(\theta) \cdot e^{-t} h_T(\theta)} = \sqrt{h_K(\theta) h_T(\theta)} \le \tfrac{h_K(\theta) + h_T(\theta)}{2} = h_{\frac{K+T}{2}}(\theta),$$

so $f_{K,T}(t) \subseteq \frac{K+T}{2}$ for every t. By monotonicity of the geometric Banach limit we also have $Lf_{K,T} \subseteq \frac{K+T}{2}$. Similarly $Lf_{T,K} \subseteq \frac{K+T}{2}$, and the right inequality follows. The left inequality follows from the right one by polarity.

Finally, we show how the scaling property (9) follows from the translation invariance of L. Write $\alpha = e^u$ and $\beta = e^v$ for some $u, v \in \mathbb{R}$. If we write $w = \frac{u+v}{2}$ and $z = \frac{u-v}{2}$ then

$$f_{\alpha K,\beta T}(t) = g\left(e^{t+u}K, e^{-t+v}T\right) = e^w g\left(e^{t+z}K, e^{-t-z}T\right) = e^w \cdot f_{K,T}(t+z).$$

Since L is homogeneous and shift invariant we have

$$Lf_{\alpha K,\beta T} = e^w Lf_{K,T} = \sqrt{\alpha\beta} Lf_{K,T}.$$

In the same way, $Lf_{\beta T,\alpha K} = \sqrt{\alpha\beta} Lf_{T,K}$, so

$$\begin{aligned} G(\alpha K, \beta T) &= g\left(Lf_{\alpha K,\beta T}, Lf_{\beta T,\alpha K}\right) = g\left(\sqrt{\alpha\beta} Lf_{K,T}, \sqrt{\alpha\beta} Lf_{T,K}\right) \\ &= \sqrt{\alpha\beta} g\left(Lf_{K,T}, Lf_{T,K}\right) = \sqrt{\alpha\beta} G(K,T) \end{aligned}$$

as we wanted. □

We now want to understand the dependence of G on the choice of a geometric Banach limit. We will need the following results on Banach limits for real-valued functions:

Lemma 11. *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $P > 0$. Then for every Banach limit $LIM : B(\mathbb{R}) \to \mathbb{R}$ one has*

$$LIM(g) = \frac{1}{P}\int_0^P g(t)dt.$$

Proof. Define a sequence $\{g_n\}_{n=1}^{\infty}$ by

$$g_n(t) = \frac{1}{n}\sum_{i=0}^{n-1} g\left(t + \frac{i}{n}P\right).$$

Since LIM is linear and shift-invariant, we have $LIM(g_n) = LIM(g)$ for all n.

Since g is continuous and periodic, it is uniformly continuous. Hence for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|t-s| < \delta$ then $|g(t) - g(s)| < \epsilon$. In particular, we see that if $n > \frac{P}{\delta}$ then for every $t \in \mathbb{R}$ we have

$$\begin{aligned} \left| g_n(t) - \frac{1}{P}\int_0^P g(s)ds \right| &= \left| \frac{1}{n}\sum_{i=0}^{n-1} g\left(t + \frac{i}{n}P\right) - \frac{1}{P}\int_t^{t+P} g(s)ds \right| \\ &= \left| \frac{1}{P}\sum_{i=0}^{n-1} \int_{t+\frac{i}{n}P}^{t+\frac{i+1}{n}P} \left(g\left(t + \frac{i}{n}P\right) - g(s)\right) ds \right| \\ &\le \frac{1}{P}\sum_{i=0}^{n-1} \int_{t+\frac{i}{n}P}^{t+\frac{i+1}{n}P} \left| g\left(t + \frac{i}{n}P\right) - g(s)\right| ds \\ &\le \frac{1}{P}\sum_{i=0}^{n-1} \int_{t+\frac{i}{n}P}^{t+\frac{i+1}{n}P} \epsilon ds = \epsilon. \end{aligned}$$

In other words, we proved that the sequence $\{g_n\}$ converges uniformly to the constant $\frac{1}{P}\int_0^P g$. Hence

$$\frac{1}{P}\int_0^P g = \lim_{n\to\infty} LIM(g_n) = LIM(g)$$

as we wanted. □

Proposition 12. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and assume $\lim_{m\to\infty} f(t + mP)$ exists uniformly in $t \in [0, P]$. Then for every Banach limit $LIM : B(\mathbb{R}) \to \mathbb{R}$ one has*

$$LIM(f) = \frac{1}{P}\int_0^P \left(\lim_{m\to\infty} f(t + mP)\right) dt.$$

Proof. First observe that if $\lim_{m\to\infty} f(t + mP)$ exists uniformly in $t \in [0, P]$ then the limit exists for all $t \in \mathbb{R}$, and the convergence is uniform on every ray of the form $[a, \infty)$.

In particular if we define $f_m(t) = f(t + mP)$ and $g(t) = \lim_{m\to\infty} f_m(t)$ then

$$LIM(f) = \lim_{m\to\infty} LIM(f_m) = LIM(g).$$

Since the function g is periodic with period P we get from the previous lemma that

$$LIM(f) = LIM(g) = \frac{1}{P}\int_0^P g,$$

which is what we wanted to prove. □

From Proposition 12 we immediately obtain the same result for convex body valued functions:

Corollary 13. *Let $f : \mathbb{R} \to \mathcal{K}^n_{(0)}$ be continuous, and assume $\lim_{m\to\infty} f(t + mP)$ exists uniformly in $t \in [0, P]$. Then for every linear Banach limit $L : \mathcal{F}^n \to \mathcal{K}^n_{(0)}$ one has*

$$L(f) = \frac{1}{P}\int_0^P \left(\lim_{m\to\infty} f(t + mP)\right) dt.$$

The integral on the right hand side should be interpreted as a Minkowski integral: $K = \int_a^b f(t)dt$ just means that for every direction θ one has $h_K(\theta) = \int_a^b h_{f(t)}(\theta)dt$.

In order to use Corollary 13, we need to prove its main assumption is satisfied in our case. We do so in the following theorem:

Theorem 14. *Fix $K, T \in \mathcal{K}^n_{(0)}$. Then the limit*

$$\lim_{m\to\infty} g\left(2^m\alpha K, \frac{1}{2^m\alpha}T\right)$$

(taken over integer values of m) exists uniformly in $\alpha \in [1, \infty)$.

Of course, the convergence is uniform not only in the interval $[1, \infty)$ but also in the interval $[\alpha_0, \infty)$ for every $\alpha_0 > 0$.

Proof. Write $Z_m(\alpha) = g\big(2^m\alpha K, \frac{1}{2^m\alpha}T\big)$. Following one step of the iteration process defining the geometric mean we also have

$$Z_m(\alpha) = g\left(\frac{2^m\alpha K + \frac{1}{2^m\alpha}T}{2}, \left(\frac{\frac{1}{2^m\alpha}K^\circ + 2^m\alpha T^\circ}{2}\right)^\circ\right).$$

Fix a number R_1 such that $T \subseteq R_1 \cdot K$, or equivalently $K^\circ \subseteq R_1 \cdot T^\circ$. Then we have

$$\begin{aligned} 2^{m-1}\alpha K \subseteq \frac{2^m\alpha K + \frac{1}{2^m\alpha}T}{2} &\subseteq \frac{2^m\alpha K + \frac{R_1}{2^m\alpha}K}{2} \\ &= \left(1 + \frac{R_1}{2^{2m}\alpha^2}\right)2^{m-1}\alpha K \subseteq \left(1 + \frac{R_1}{2^{2m}}\right)2^{m-1}\alpha K. \end{aligned}$$

In exactly the same way, we see that

$$2^{m-1}\alpha T^\circ \subseteq \frac{\frac{1}{2^m\alpha}K^\circ + 2^m\alpha T^\circ}{2} \subseteq \left(1+\frac{R_1}{2^{2m}}\right)2^{m-1}\alpha T^\circ,$$

which after duality is equivalent to

$$\frac{1}{1+\frac{R_1}{2^{2m}}}\cdot\frac{1}{2^{m-1}\alpha}T \subseteq \left(\frac{\frac{1}{2^m\alpha}K^\circ + 2^m\alpha T^\circ}{2}\right)^\circ \subseteq \frac{1}{2^{m-1}\alpha}T.$$

It follows that we have

$$\begin{aligned} Z_m(\alpha) &\subseteq g\left(\left(1+\frac{R_1}{2^{2m}}\right)2^{m-1}\alpha K, \frac{1}{2^{m-1}\alpha}T\right) \\ &\subseteq \left(1+\frac{R_1}{2^{2m}}\right) g\left(2^{m-1}\alpha K, \frac{1}{2^{m-1}\alpha}T\right) = \left(1+\frac{R_1}{2^{2m}}\right) Z_{m-1}(\alpha), \end{aligned}$$

and similarly

$$Z_m(\alpha) \supseteq \left(1+\frac{R_1}{2^{2m}}\right)^{-1}\cdot Z_{m-1}(\alpha).$$

In order to pass from these inclusions to the Hausdorff distance, we need to choose a number R_2 large enough to have $K, T \subseteq R_2\cdot B_2^n$, which implies that $Z_m(\alpha) \subseteq R_2\cdot B_2^n$ for all m and all α. Hence

$$Z_m(\alpha) \subseteq Z_{m-1}(\alpha) + \frac{R_1R_2}{2^{2m}}B_2^n$$

and

$$Z_{m-1}(\alpha) \subseteq Z_m(\alpha) + \frac{R_1R_2}{2^{2m}}B_2^n,$$

so we have the uniform estimate $d_H\left(Z_{m-1}(\alpha), Z_m(\alpha)\right) \le \frac{R_1R_2}{2^{2m}}$. Since $\sum_{m=1}^\infty \frac{R_1R_2}{2^{2m}}$ converges, it follows that $\{Z_m(\alpha)\}$ is uniformly Cauchy.

It follows that $Z_m(\alpha) \to Z(\alpha)$ uniformly in $\alpha \in [1,\infty)$. We do not know a priori that $Z(\alpha) \in \mathcal{K}^n_{(0)}$, as it may have an empty interior. However, if we choose $\epsilon > 0$ such that $K, T \supseteq \epsilon\cdot B_2^n$ then $Z_m(\alpha) \supseteq \epsilon B_2^n$ for all m and all α, which implies that $Z(\alpha) \supseteq \epsilon B_2^n$. Hence $Z(\alpha) \in \mathcal{K}^n_{(0)}$, and the proof is complete. □

Putting everything together, we have the following result, which shows that the construction of $G(K,T)$ can be written explicitly without referring to Banach limits:

Proposition 15. *For $K, T \in \mathcal{K}^n_{(0)}$, define $f_{K,T} \in \mathcal{F}^n$ by $f_{K,T}(t) = g(e^tK, e^{-t}T)$. Then for every linear Banach limit on $\mathcal{F}^n$ one has*

$$Lf_{K,T} = \int_0^1 \left(\lim_{m\to\infty} g\left(2^{m+t}K, 2^{-m-t}T\right)\right)dt.$$

Proof. We will check that the condition of Corollary 13 is satisfied with $P = \ln 2$. Indeed,

$$\lim_{m\to\infty} f_{K,T}(t + m\ln 2) = \lim_{m\to\infty} g\left(2^me^tK, 2^{-m}e^{-t}T\right),$$

and the existence of this limit is ensured by Theorem 14. Furthermore, by the same theorem the limit is uniform in t as long as $e^t \ge 1$, or equivalently $t \in [0,\infty)$. Hence we may apply Corollary 13 and deduce that

$$Lf_{K,T} = \frac{1}{\ln 2}\int_0^{\ln 2}\left(\lim_{m\to\infty} g\left(2^me^tK, 2^{-m}e^{-t}T\right)\right)dt.$$

The change of variables $t = s\ln 2$ gives the result. □

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