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A sharp Blaschke–Santaló inequality for α-concave functions

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Abstract We define a new transform on α-concave functions, which we call the $\sharp$-transform. Using this new transform, we prove a sharp Blaschke-Santaló inequality for α-concave functions, and characterize the equality case. This extends the known functional Blaschke-Santaló inequality of Artstein-Avidan, Klartag and Milman, and strengthens a result of Bobkov.

Finally, we prove that the $\sharp$-transform is a duality transform when restricted to its image. However, this transform is neither surjective nor injective on the entire class of α-concave functions.

Keywords Blaschke-Santaló inequality · convexity · α-concavity · log-concavity

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1 Blaschke–Santaló type inequalities

We begin by recalling the classic Blaschke-Santaló inequality. A *convex body* in $\mathbb{R}^n$ is compact, convex set with non-empty interior. Such a convex body K is called *symmetric* if $K = -K$. We will denote by $|K|$ the (Lebesgue) volume of K. Finally, we define the *polar body* of K to be

$$K^\circ = \{x \in \mathbb{R}^n : \ \langle x, y \rangle \leq 1 \text{ for all } y \in K\},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^n$.

Polarity is a basic notion in convex geometry. It is easy to see that if K is symmetric, convex body, then so is K°. The map $K \mapsto K^\circ$ satisfies two fundamental properties:

1. It is *order reversing*: If $K_1 \subseteq K_2$, then $K_1^\circ \supseteq K_2^\circ$.
2. It is an *involution*: For every symmetric convex body K we have $K = (K^\circ)^\circ$.

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These two properties say that polarity is a *duality transform* (see, e.g., [2]).

The volume of K° is related to the volume of K by the Blaschke-Santaló inequality

Theorem 1 (Blaschke, Santaló) *Assume $K \subseteq \mathbb{R}^n$ is a symmetric, convex body. Then*

$$|K| \cdot \left|K^\circ\right| \leq |D|^2 ,$$

where $D \subseteq \mathbb{R}^n$ is the unit Euclidean ball. Furthermore, we have an equality if and only if K is an ellipsoid.

This theorem was proven by Blaschke in dimensions 2 and 3, and Santaló extended the result to arbitrary dimensions. There exists a version of the inequality for non-symmetric bodies, but for simplicity we will only deal with the symmetric case. The generalized statement, proofs and further references can be found, e.g., in [12].

In [1], Artstein-Avidan, Klartag and Milman prove a functional extension of the Blaschke-Santaló inequality. To explain their result and put it in perspective, we will need to define α-concave functions:

Definition 1 Fix $-\infty \leq \alpha \leq \infty$. We say that a function $f : \mathbb{R}^n \to [0, \infty)$ is α-concave if f is supported on some convex set Ω, and for every $x, y \in \Omega$ and $0 \leq \lambda \leq 1$ we have

$$f\left(\lambda x + (1-\lambda)y\right) \geq \left[\lambda f(x)^\alpha + (1-\lambda)\, f(y)^\alpha\right]^{\frac{1}{\alpha}} .$$

We will always assume that f is upper semicontinuous and that

$$\max_{x \in \mathbb{R}^n} f(x) = f(0) = 1$$

(this last condition is sometimes known as saying that f is *geometric*).

The class of all such α-concave functions will be denoted by $\mathrm{C}_\alpha\left(\mathbb{R}^n\right)$.

Remember that $f : \mathbb{R}^n \to [0, \infty)$ is called upper semicontinuous if its upper level sets $\{x \in \mathbb{R}^n : \ f(x) \geq t\}$ are closed for all $t \geq 0$.

α-concave functions were first defined by Avriel ([5]), and were studied by Borell ([8],[9]) and by Brascamp and Lieb ([10]). In the case $\alpha = \infty$ we understand the definition in the limiting sense as

$$f\left(\lambda x + (1-\lambda)y\right) \geq \max\left\{f(x), f(y)\right\} .$$

This just means that f is constant on its support, so functions in $\mathrm{C}_\infty\left(\mathbb{R}^n\right)$ are indicator functions of convex sets, which by our assumptions must also be closed and contain the origin. If we further assume that $f \in \mathrm{C}_\infty\left(\mathbb{R}^n\right)$ satisfy $0 < \int f < \infty$, then f must be the indicator function of a convex body, and we can identify these functions with the convex bodies themselves.

If $\alpha_1 < \alpha_2$, then $\mathrm{C}_{\alpha_1}\left(\mathbb{R}^n\right) \supseteq \mathrm{C}_{\alpha_2}\left(\mathbb{R}^n\right)$. This means that we can think of every class of functions $\mathrm{C}_\alpha\left(\mathbb{R}^n\right)$ as extending the class of convex bodies. Originally, this was done for the class $\mathrm{C}_0\left(\mathbb{R}^n\right)$ of *log-concave* functions. Again we interpret Definition 1 in the limiting sense, and say that a function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

$$f\left(\lambda x + (1-\lambda)y\right) \geq f(x)^\lambda f(y)^{1-\lambda}.$$

for every $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$.

Many definitions and theorems of convex geometry were generalized to the class of log-concave functions. Except the usual aspiration for generality, the developed theory helped to prove new deep theorems in convexity and asymptotic geometric analysis. For a survey of such results and their importance, see [16].

One of the first results in this new direction was the functional Santaló theorem. In order to state it we will begin with a convenient definition:

Definition 2 For every $f \in \mathrm{C}_0(\mathbb{R}^n)$ we define

$$f^* = \exp\left(-\mathcal{L}\left(-\log f\right)\right) \in \mathrm{C}_0\left(\mathbb{R}^n\right),$$

where $\mathcal{L}$ is the Legendre transform, defined by

$$\left(\mathcal{L}\varphi\right)(x) = \sup_{y\in\mathbb{R}^n}\left(\langle x, y\rangle - \varphi(y)\right).$$

This definition just means that if $f = e^{-\varphi}$ for a convex function φ, then $f^* = e^{-\mathcal{L}\varphi}$. It turns out that the map $f \mapsto f^*$ is a duality transform on $\mathrm{C}_0(\mathbb{R}^n)$, and we have the following theorem:

Theorem 2 *Assume $f \in \mathrm{C}_0(\mathbb{R}^n)$ is even and $0 < \int f < \infty$. Then*

$$\int f \cdot \int f^* \le \left(\int G\right)^2 = (2\pi)^n,$$

where

$$G(x) = e^{-\frac{|x|^2}{2}}.$$

Equality occurs if and only if $f = G \circ T$ for an invertible linear map $T : \mathbb{R}^n \to \mathbb{R}^n$.

This inequality, in the even case, is originally due to Ball ([6]). In [1], Artstein-Avidan, Klartag and Milman present this result as a Blaschke–Santaló type inequality, extend the result to the non-even case and characterize the equality case.

As a side note, let us mention that given our current knowledge, the form of Theorem 2 is somewhat surprising. Following a series of works by Artstein-Avidan and Milman, we now understand that even though the map $f \mapsto f^*$ is a duality transform, it is *not* the correct extension of the classic notion of polarity (see [3]). The correct extension is another duality transform, usually denoted $f \mapsto f^\circ$, which is based on the so-called $\mathcal{A}$-transform. Hence we expect the functional Blaschke–Santaló inequality to bound an expression of the form

$$\int f \cdot \int f^\circ, \tag{1}$$

which is not what we see in Theorem 2. In fact, a sharp upper bound on (1) is not known, even though an asymptotic result was recently found by Artstein-Avidan and Slomka ([4]).

The goal of this paper is to discuss Blaschke-Santaló inequalities for α-concave functions, for values of α different from 0. The case $\alpha > 0$ was resolved already in [1], so we will assume from now on that $\alpha \le 0$. The following definition appeared in [17]:

Definition 3 Assume $-\infty < \alpha < 0$. The convex base of a function $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ is

$$\mathrm{base}_\alpha f = \frac{1 - f^\alpha}{\alpha}.$$

Put differently, $\varphi = \mathrm{base}_\alpha f$ is the unique convex function such that

$$f = \left(1 + \frac{\varphi}{\beta}\right)^{-\beta}.$$

Here and after, we use the parameter $\beta = -\frac{1}{\alpha}$. As $\alpha \le 0$ and $\beta \ge 0$, it is often less confusing to use β instead of α. In the limiting case $\alpha = 0$ we define $\mathrm{base}_0(f) = -\log f$. Using the notion of a convex base we can extend Definition 2 to the general case:

Definition 4 The dual of a function $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ is the function $f^* \in \mathrm{C}_\alpha(\mathbb{R}^n)$ defined by relation

$$\mathrm{base}_\alpha(f^*) = \mathcal{L}(\mathrm{base}_\alpha(f)).$$

Note that the operation $*$ depends on α. Remember that if $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ then $f \in \mathrm{C}_{\alpha'}(\mathbb{R}^n)$ for all $\alpha' < \alpha$. Thinking of f as an element in $\mathrm{C}_{\alpha'}(\mathbb{R}^n)$ will yield a different f^* than thinking about f as function in $\mathrm{C}_\alpha(\mathbb{R}^n)$. Therefore, strictly speaking, we should use a notation like f^{*_α}. However, this notation is extremely cumbersome, so we will use the simpler notation f^*, and keep in mind the implicit dependence on α.

We can now state what appears to be the natural extension of Theorem 2 to the α-concave case:

Theorem 3 *Fix* $-\frac{1}{n} < \alpha \le 0$. *For every even function* $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ *such that* $0 < \int f < \infty$ *we have*

$$\int f \cdot \int f^* \le \left(\int H_\alpha\right)^2, \tag{2}$$

where

$$H_\alpha(x) = \left(1 + \frac{|x|^2}{\beta}\right)^{-\frac{\beta}{2}} \in \mathrm{C}_\alpha(\mathbb{R}^n).$$

This theorem was proven by Bobkov ([7]) using a general result by Fradelizi and Meyer ([11]), which we will cite in the next section as Theorem 5. The condition $\alpha > -\frac{1}{n}$ is necessary, because the function H_α is no longer integrable for $\alpha \le -\frac{1}{n}$. In fact, it is not hard to check that

$$\int H_\alpha \cdot \int H_\alpha^* \to \infty$$

as $\alpha \to -\frac{1}{n}$, so it is not possible to find a finite upper bound for $\int f \cdot \int f^*$ whenever $\alpha \le -\frac{1}{n}$. Notice that as $\alpha \to 0$ the functions H_α converge to the Gaussian G, and we obtain Theorem 2 as a special case of Theorem 3.

Surprisingly, as was already observed by Bobkov, Theorem 3 is not sharp when $\alpha < 0$. Given the previously described results, it is very natural to expect an equality in (2) when $f = H_\alpha$. However, an explicit (yet tedious) calculation shows that $H_\alpha^*(x) < H_\alpha(x)$ for all $x \neq 0$, so

$$\int H_\alpha \cdot \int H_\alpha^* < \left(\int H_\alpha\right)^2.$$

In fact, there exists a unique function $G_\alpha \in \mathrm{C}_\alpha(\mathbb{R}^n)$ such that $G_\alpha^* = G_\alpha$, and this function is

$$G_\alpha(x) = \left(1 + \frac{|x|^2}{2\beta}\right)^{-\beta}.$$

Notice that in the limiting case $\alpha \to 0$ we have $H_0 = G_0 = G$, but for other values of α these functions are quite different. Inspired by Theorems 1 and 2, it may seem reasonable to conjecture that

$$\int f \cdot \int f^* \le \left(\int G_\alpha\right)^2$$

for all even $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$. Such an inequality, if true, will obviously be sharp. Unfortunately, this inequality is false for $\alpha < 0$. As one possible counterexample, take $f = \mathbf{1}_D$, the indicator function of the ball. In this case

$$f^* = \left(1 + \frac{|x|}{\beta}\right)^{-\beta},$$

and a direct computation of the integrals show that this is indeed a counterexample if the dimension n is large enough compared to β (For concreteness, it is enough to take $n = \left\lceil \frac{\beta}{2} \right\rceil$ for all large enough β).

In the next sections we will show the reason inequality (2) is not sharp is that the transform $*$ is not the correct extension of polarity to use in the functional Blaschke-Santaló inequality. In section 2 we will define a new transform on $\mathrm{C}_\alpha(\mathbb{R}^n)$, which we call $\sharp$-transform. We will then use this $\sharp$-transform to prove a sharp version of Theorem 3. Finally, in section 3, we will discuss further properties of $\sharp$ which are not directly related to the Blaschke-Santaló inequality, and give a geometric interpretation of this transform.

2 A new transform on α-concave functions

In [1], Artstein-Avidan, Klartag and Milman obtain Theorem 2 as the limit of Blaschke-Santaló type inequalities for α-concave functions, $\alpha \ge 0$. Let us warn the reader that [1] uses a slightly different notation than the one used in this paper: an α-concave function in this paper is the same as a $\frac{1}{\alpha}$-concave function in [1], and vice versa.

Inspired by the transforms in [1], we define the following:

Definition 5 For $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ we define

$$f^\sharp(x) = \inf_y \frac{1}{f(y) \cdot \left(1 + \frac{\langle x,y \rangle}{\beta}\right)^\beta} = \frac{1}{\sup_y \left[f(y) \cdot \left(1 + \frac{\langle x,y \rangle}{\beta}\right)^\beta\right]},$$

where the infimum is taken over all points $y \in \mathbb{R}^n$ such that $f(y) > 0$ and $\langle x, y \rangle > -\beta$.

Like the $*$ transform, the $\sharp$ transform also depends on α, so in principle we should write $f^{\sharp_\alpha}$. Nonetheless, we opt for the simpler notation $f^\sharp$.

Let us begin by checking that $f^\sharp \in \mathrm{C}_\alpha(\mathbb{R}^n)$:

Proposition 1 *For every $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ we have $f^\sharp \in \mathrm{C}_\alpha(\mathbb{R}^n)$. If f is even, so is $f^\sharp$.*

Proof For a fixed $y \in \mathbb{R}^n$ with $f(y) > 0$ the function

$$f_y(x) = \begin{cases} \frac{1}{f(y)} \cdot \left(1 + \frac{\langle x,y \rangle}{\beta}\right)^{-\beta} & \text{if } \langle x, y \rangle > -\beta, \\ \infty & \text{otherwise} \end{cases}$$

is upper semicontinuous and α-concave (except the fact it can attain the value $+\infty$, which we usually exclude from the definition). Now we can write

$$f^\sharp(x) = \inf_{y:\ f(y)>0} f_y(x),$$

so $f^\sharp$ is α-concave as the infimum of a family of α-concave functions. Similarly $f^\sharp$ is upper semicontinuous, as the infimum of a family of upper semicontinuous functions.

For every $x \in \mathbb{R}^n$ we have

$$f^\sharp(x) = \inf_y \frac{1}{f(y) \cdot \left(1 + \frac{\langle x,y \rangle}{\beta}\right)^\beta} \le \frac{1}{f(0)\left(1 + \frac{\langle x,0 \rangle}{\beta}\right)^\beta} = 1.$$

Additionally,

$$f^\sharp(0) = \inf_y \frac{1}{f(y)\left(1 + \frac{\langle 0,y \rangle}{\beta}\right)_+^\beta} = \inf_y \frac{1}{f(y)} = \frac{1}{\sup_y f(y)} = 1.$$

and we see that $f^\sharp$ is geometric. Hence we have $f^\sharp \in \mathrm{C}_\alpha(\mathbb{R}^n)$ like we wanted.

Finally if f is even then

$$\begin{aligned} f^\sharp(-x) &= \inf_y \frac{1}{f(y) \cdot \left(1 + \frac{\langle -x,y \rangle}{\beta}\right)^\beta} = \inf_y \frac{1}{f(-y) \cdot \left(1 + \frac{\langle -x,-y \rangle}{\beta}\right)^\beta}, \\ &= \inf_y \frac{1}{f(y) \cdot \left(1 + \frac{\langle x,y \rangle}{\beta}\right)^\beta} = f^\sharp(x), \end{aligned}$$

so $f^\sharp$ is even as well.

Our main goal in this section is to prove the following theorem:

Theorem 4 *Fix $-\frac{1}{n} < \alpha \le 0$. For every even $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ such that $0 < \int f < \infty$ we have*

$$\int f \cdot \int f^\sharp \le \left(\int H_\alpha\right)^2,$$

with equality if and only if $f = H_\alpha \circ T$ for an invertible linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$.

Here we have $H_\alpha(x) = \left(1 + \frac{|x|^2}{\beta}\right)^{-\frac{\beta}{2}}$, just like in Theorem 3 of Bobkov. For $\alpha \le -\frac{1}{n}$ one cannot hope for a finite upper bound on the product $\int f \cdot \int f^\sharp$. This can be seen by choosing $f = H_\alpha$, and considering the following proposition:

Proposition 2 *$H_\alpha^\sharp = H_\alpha$, and H_α is the only function with this property.*

Proof First we calculate $H_\alpha^\sharp$ explicitly and show that $H_\alpha^\sharp = H_\alpha$. By definition,

$$H_\alpha^\sharp(x) = \left[\sup_y H_\alpha(y)\left(1+\frac{\langle x,y\rangle}{\beta}\right)^\beta\right]^{-1}$$
$$= \left[\sup_y \underbrace{\left(1+\frac{|y|^2}{\beta}\right)^{-\frac{\beta}{2}}\left(1+\frac{\langle x,y\rangle}{\beta}\right)^\beta}_{(\star)}\right]^{-1}.$$

Notice that if we take a vector y and rotate it to have the same direction as x, we can only increase the expression $(\star)$. Hence

$$H_\alpha^\sharp(x) = \left[\sup_{\lambda>0}\left(1+\frac{|\lambda x|^2}{\beta}\right)^{-\frac{\beta}{2}}\left(1+\frac{\langle x,\lambda x\rangle}{\beta}\right)^\beta\right]^{-1}$$
$$= \left[\sup_{\lambda>0}\frac{\left(1+\frac{\lambda|x|^2}{\beta}\right)^2}{1+\frac{\lambda^2|x|^2}{\beta}}\right]^{-\frac{\beta}{2}}.$$

It is now an exercise in calculus to differentiate and check that the supremum is actually a maximum, which is obtained for $\lambda = 1$. Hence

$$H_\alpha^\sharp(x) = \left[\frac{\left(1+\frac{|x|^2}{\beta}\right)^2}{1+\frac{|x|^2}{\beta}}\right]^{-\frac{\beta}{2}} = \left(1+\frac{|x|^2}{\beta}\right)^{-\frac{\beta}{2}} = H_\alpha(x)$$

which is what we needed to show.

Now assume that $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ is any function such that $f^\sharp = f$. For every $x \in \mathbb{R}^n$ with $f(x) > 0$ we have

$$f(x) = f^\sharp(x) = \inf_y \frac{1}{f(y)\cdot\left(1+\frac{\langle x,y\rangle}{\beta}\right)^\beta} \le \frac{1}{f(x)\cdot\left(1+\frac{|x|^2}{\beta}\right)^\beta},$$

so multiplying by $f(x)$ and taking a square root we get $f(x) \le H_\alpha(x)$. If $f(x) = 0$ then $f(x) \le H_\alpha(x)$ holds trivially, so the inequality is true for all $x \in \mathbb{R}^n$.

It is obvious from the definition that $\sharp$ is order reversing, so we may apply it on both sides and obtain

$$f = f^\sharp \ge H_\alpha^\sharp = H_\alpha,$$

so $f = H_\alpha$ like we wanted.

Theorem 4, like Theorem 3, will follow from a general result of Fradelizi and Meyer ([11]). We state their result here:

Theorem 5 *Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}_+$ and $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions, such that*

$$f_1(x) f_2(y) \le \rho^2(\langle x, y\rangle)$$

for every $x, y \in \mathbb{R}^n$ such that $\langle x, y\rangle > 0$. If, additionally, f_1 is even, then

$$\int f_1 \cdot \int f_2 \le \left(\int \rho(|x|^2) dx \right)^2 .$$

Assume further that ρ is continuous. Then equality will occur if and only if:

1. *$\sqrt{\rho(s)\rho(t)} \le \rho(\sqrt{st})$ for every $s, t \ge 0$.*
2. *If $n \ge 2$ then either $\rho(0) > 0$ or $\rho \equiv 0$.*
3. *There exists a positive definite matrix T and a constant $d > 0$ such that*

$$f_1(x) = d \cdot \rho\left(|Tx|^2\right), \quad f_2(x) = \frac{1}{d} \cdot \rho\left(\left|T^{-1}x\right|^2\right)$$

almost everywhere.

Let us use this result to prove Theorem 4:

Proof (Proof of Theorem 4) Define a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\rho(t) = \left(1 + \frac{t}{\beta}\right)^{-\beta/2} .$$

Fix $x, y \in \mathbb{R}^n$ with $\langle x, y\rangle > 0$. If $f(x) = 0$ then obviously $f(x) f^\sharp(y) \le \rho^2(\langle x, y\rangle)$. If, on the other hand, $f(x) > 0$ then

$$\begin{aligned} f(x) \cdot f^\sharp(y) &= \inf_z \frac{f(x)}{f(z) \cdot \left(1 + \frac{\langle y, z\rangle}{\beta}\right)_+^\beta} \le \frac{f(x)}{f(x) \cdot \left(1 + \frac{\langle y, x\rangle}{\beta}\right)^\beta} \\ &= \left(1 + \frac{\langle x, y\rangle}{\beta}\right)^{-\beta} = \rho^2(\langle x, y\rangle) . \end{aligned}$$

From Theorem 5 we conclude that indeed

$$\int f \cdot \int f^\sharp \le \left(\int \rho(|x|^2) dx \right)^2 = \left(\int H_\alpha \right)^2 .$$

Next we analyze the equality case. From Theorem 5 we see that a necessary condition to have equality is

$$f(x) = d \cdot \rho\left(|Tx|^2\right) = d \cdot H_\alpha(Tx)$$

for a constant $d > 0$ and a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ (which we may take to be positive definite if we want). Since $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ we know that

$$1 = f(0) = d \cdot H_\alpha(0) = d \cdot 1 = d,$$

so we must have $f = H_\alpha \circ T$.

To see that this condition is also sufficient, notice that for every $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ and every invertible linear map T we have

$$(f \circ T)^\sharp(x) = \left[\sup_y f(Ty)\left(1+\frac{\langle x,y\rangle}{\beta}\right)^\beta\right]^{-1} = \left[\sup_y f(y)\left(1+\frac{\langle x,T^{-1}y\rangle}{\beta}\right)^\beta\right]^{-1}$$
$$= \left[\sup_y f(y)\left(1+\frac{\left\langle \left(T^{-1}\right)^* x,y\right\rangle}{\beta}\right)^\beta\right]^{-1} = f^\sharp\left(\left(T^{-1}\right)^* x\right).$$

Using a simple change of variables and Proposition 2 we get that if $f = H_\alpha \circ T$ then

$$\int f \cdot \int f^\sharp = \int (H_\alpha \circ T) \cdot \int \left(H_\alpha^\sharp \circ \left(T^{-1}\right)^*\right)$$
$$= \frac{1}{\det(T)\cdot\det\left(\left(T^{-1}\right)^*\right)}\int H_\alpha \int H_\alpha^\sharp = \left(\int H_\alpha\right)^2$$

so we are done.

Remark 1 For simplicity, Theorem 4 is only stated for even functions. Fradelizi and Meyer also proved in [11] a generalization of Theorem 5 for non-even functions, which can be used to extend Theorem 4 to the non-even case. The proof remains essentially the same, so we leave the details to the interested reader.

To conclude this section, let us compare Theorem 4 with Theorem 3. We have the following proposition:

Proposition 3 *For every* $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ *we have* $f^\sharp \ge f^*$.

Proof Denote $\varphi = \mathrm{base}_\alpha(f)$. We need to prove that for every $x \in \mathbb{R}^n$ we have $f^\sharp(x) \ge f^*(x)$, which is equivalent to

$$\inf_y \left(\frac{1+\frac{\langle x,y\rangle}{\beta}}{1+\frac{\varphi(y)}{\beta}}\right)^{-\beta} \ge \inf_y \left(1+\frac{\langle x,y\rangle-\varphi(y)}{\beta}\right)^{-\beta}.$$

Choose a sequence $\{y_n\}$ such that

$$\left(\frac{1+\frac{\langle x,y_n\rangle}{\beta}}{1+\frac{\varphi(y_n)}{\beta}}\right)^{-\beta} \to f^\sharp(x).$$

We claim it is always possible to choose this sequence in such a way that $\langle x,y_n\rangle \ge \varphi(y_n) \ge 0$ for every n. Indeed, we know that $f^\sharp(x) \le 1$. If $f^\sharp(x) < 1$ we will automatically have $\langle x,y_n\rangle > \varphi(y_n) \ge 0$ for large enough n. If $f^\sharp(x) = 1$, just take $y_n = 0$ for all n.

For every two numbers $B \ge A \ge 0$ we have

$$\frac{1+B}{1+A} \le 1+B-A,$$

as one easily checks. Applying this to $B = \frac{\langle x, y_n\rangle}{\beta}$ and $A = \frac{\varphi(y_n)}{\beta}$ we see that

$$\left(\frac{1+\frac{\langle x,y_n\rangle}{\beta}}{1+\frac{\varphi(y_n)}{\beta}}\right)^{-\beta} \ge \left(1+\frac{\langle x, y_n\rangle - \varphi(y_n)}{\beta}\right)^{-\beta} \ge f^*(x)$$

for all n. Sending $n \to \infty$ we see that $f^\sharp(x) \ge f^*(x)$ like we wanted.

This means that for every value of α Theorem 3 follows from Theorem 4. When $\alpha \to 0$ the transforms $*$ and $\sharp$ coincide, so both theorems reduce to same result - Theorem 2.

3 Further properties of $\sharp$-transform

In this section we will discuss further properties of the new $\sharp$-transform, $\sharp : \mathrm{C}_\alpha(\mathbb{R}^n) \to \mathrm{C}_\alpha(\mathbb{R}^n)$ for $\alpha < 0$. We already used the simple fact that $\sharp$ is order reversing: if $f, g \in \mathrm{C}_\alpha(\mathbb{R}^n)$ and $f \le g$ (pointwise), then $f^\sharp \ge g^\sharp$. Surprisingly, however, $\sharp$ is *not* a duality transform, as it is not an involution.

One simple way of verifying the last assertion is by computing a few examples:

Example 1 Let K be a convex body containing the origin. Remember that the gauge function of K is defined by

$$\|x\|_K = \inf\left\{r > 0 : \ \frac{x}{r} \in K\right\}.$$

Let us denote by $\mathbf{1}_K$ the indicator function of K. Then a simple calculation gives

$$\mathbf{1}_K^\sharp = \mathbf{1}_K^* = \left(1 + \frac{\|x\|_{K^\circ}}{\beta}\right)^{-\beta},$$

and

$$\left[\left(1+\frac{\|x\|_K}{\beta}\right)^{-\beta}\right]^\sharp = \min\left\{\frac{1}{\|x\|_{K^\circ}^\beta}, 1\right\}.$$

In particular, we see that $\mathbf{1}_K^{\sharp\sharp} \ne \mathbf{1}_K$ for every $\alpha < 0$ (equivalently, for every $\beta < \infty$).

In order to prove more delicate properties of the $\sharp$-transform, we will need to examine it from a different point view. Denote by $\mathrm{Cvx}_0(\mathbb{R}^n)$ the class of all convex functions $\varphi : \mathbb{R}^n \to [0, \infty]$ such that φ is lower semicontinuous and $\varphi(0) = 0$. The map $\mathrm{base}_\alpha : \mathrm{C}_\alpha(\mathbb{R}^n) \to \mathrm{Cvx}_0(\mathbb{R}^n)$ from Definition 3 is easily seen to be an order reversing bijection. Hence, if we wish to understand the $\sharp$-transform, it is enough to study its conjugate $\mathcal{T}_\alpha : \mathrm{Cvx}_0(\mathbb{R}^n) \to \mathrm{Cvx}_0(\mathbb{R}^n)$ which is defined by

$$\mathcal{T}_\alpha = \mathrm{base}_\alpha \circ \sharp \circ \left(\mathrm{base}_\alpha^{-1}\right).$$

The transform $\mathcal{T}_\alpha$ can be written down explicitly:

Proposition 4 *For every $\varphi \in \mathrm{Cvx}_0(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$ we have*

$$(\mathcal{T}_\alpha\varphi)(x) = \sup_{y\in\mathbb{R}^n} \frac{\langle x, y\rangle - \varphi(y)}{1-\alpha\varphi(y)} = \sup_{y\in\mathbb{R}^n} \frac{\langle x, y\rangle - \varphi(y)}{1+\frac{\varphi(y)}{\beta}}. \tag{3}$$

In particular we see that $\mathcal{T}_0 = \mathcal{L}$ is the Legendre transform. This also follows from the fact that on $\mathrm{C}_0(\mathbb{R}^n)$ the $\sharp$-transform and the $*$-transform coincide.

Proof Let us use equation (3) as the definition of $\mathcal{T}_\alpha$, and check that under this definition we really have

$$\mathcal{T}_\alpha = \mathrm{base}_\alpha \circ \sharp \circ \left(\mathrm{base}_\alpha^{-1}\right).$$

This is of course the same as $\left(\mathrm{base}_\alpha^{-1}\right) \circ \mathcal{T}_\alpha = \sharp \circ \left(\mathrm{base}_\alpha^{-1}\right)$. Plugging in all of the definitions, we need to prove that for every $\varphi \in \mathrm{Cvx}_0(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$

$$\left(1 + \frac{1}{\beta} \cdot \sup_y \frac{\langle x, y\rangle - \varphi(y)}{1 + \frac{\varphi(y)}{\beta}}\right)^{-\beta} = \inf_y \frac{\left(1 + \frac{\langle x,y\rangle}{\beta}\right)^{-\beta}}{\left(1 + \frac{\varphi(y)}{\beta}\right)^{-\beta}},$$

and checking this equality involves nothing more than simple algebra.

Interestingly, the transforms $\mathcal{T}_\alpha$ were introduced and studied by Milman around 1970 for very different applications in functional analysis (see [13], [15] for the original papers in Russian and section 3.3 of the survey [14] for a partial translation to English. The remark in the end of section 3 of [1] is also relevant, but inaccurate). The only result we will need from these works is the following geometric characterization of $\mathcal{T}_\alpha$:

Fix a function $\varphi \in \mathrm{Cvx}_0(\mathbb{R}^n)$. We will use φ to construct a function $\rho : \mathbb{R}^n \times \mathbb{R} \to [0, \infty]$ in the following way: first, we define

$$\rho\left(x, \sqrt{\beta}\right) = \frac{\beta + \varphi(x)}{\sqrt{\beta}}.$$

Next, we extend ρ by requiring it to be 1-homoegeneous. Hence for every $x \in \mathbb{R}^n$ and $t \neq 0$ we define

$$\rho(x,t) = \rho\left(\frac{t}{\sqrt{\beta}} \cdot \left(\frac{x\sqrt{\beta}}{t}, \sqrt{\beta}\right)\right) = \frac{|t|}{\sqrt{\beta}} \cdot \frac{\beta + \varphi\left(\frac{x\sqrt{\beta}}{t}\right)}{\sqrt{\beta}} = |t| + \frac{|t|}{\beta}\varphi\left(\frac{x\sqrt{\beta}}{t}\right).$$

The values of ρ on the hyperplane $t = 0$ are not so important, but for concreteness we will define $\rho(x, 0) = \lim_{t\to 0^+} \rho(x,t)$ (the limit exists by the convexity of φ).

The function ρ is 1-homoegeneous by construction, but in general it will *not* be a norm on $\mathbb{R}^{n+1}$, since there is no reason for ρ to be convex. Nonetheless, we can define the "dual" norm

$$\rho^*(x,t) = \sup_{(y,s)\in\mathbb{R}^{n+1}} \frac{\langle x, y\rangle + ts}{\rho(y,s)},$$

which is always a proper norm on $\mathbb{R}^{n+1}$. Now if we restrict ourselves back to the hyperplane $t = \beta$ a direct calculation gives

$$\rho^*(x, \sqrt{\beta}) = \frac{\beta + (\mathcal{T}_\alpha\varphi)(x)}{\sqrt{\beta}},$$

which shows the relation between the transform $\mathcal{T}_\alpha$ and the classic notion of duality.

Using this construction we can prove several properties of the $\sharp$-transform. Specifically we have:

Theorem 6 *Fix $-\infty < \alpha < 0$, and let $\sharp : \mathrm{C}_\alpha(\mathbb{R}^n) \to \mathrm{C}_\alpha(\mathbb{R}^n)$ be the $\sharp$-transform. Then:*

1. *For every $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ we have $f^{\sharp\sharp} \ge f$.*
2. *For every $f \in \mathrm{C}_\alpha(\mathbb{R}^n)$ we have $f^{\sharp\sharp\sharp} = f^\sharp$. In other words, $\sharp$ is a duality transform on its image.*
3. *$\sharp$ is neither injective nor surjective.*

Theorem 6 is an immediate corollary of the following proposition, establishing the same properties for the transform $\mathcal{T}_\alpha$:

Proposition 5 *Fix $-\infty < \alpha < 0$, and let $\mathcal{T}_\alpha : \mathrm{Cvx}_0(\mathbb{R}^n) \to \mathrm{Cvx}_0(\mathbb{R}^n)$ be the transform defined above. Then:*

1. *For every $\varphi \in \mathrm{Cvx}_0(\mathbb{R}^n)$ we have $\mathcal{T}_\alpha^2\varphi \le \varphi$.*
2. *For every $\varphi \in \mathrm{Cvx}_0(\mathbb{R}^n)$ we have $\mathcal{T}_\alpha^3\varphi = \mathcal{T}_\alpha\varphi$. In other words, $\mathcal{T}_\alpha$ is a duality transform on its image.*
3. *$\mathcal{T}_\alpha$ is neither injective nor surjective.*

Proof Fix $\varphi \in \mathrm{Cvx}_0(\mathbb{R}^n)$, and let $\rho : \mathbb{R}^{n+1} \to [0,\infty]$ we defined as above. It is well known that if ρ is any 1-homogenous function, which is not necessarily convex, then $\rho^{**} \le \rho$. In particular

$$\frac{\beta + \left(\mathcal{T}_\alpha^2\varphi\right)(x)}{\sqrt{\beta}} = \rho^{**}(x,\sqrt{\beta}) \le \rho(x,\sqrt{\beta}) = \frac{\beta + \varphi(x)}{\sqrt{\beta}},$$

which proves (1).

Since ρ^* is already a norm, we must have $\rho^{***} = (\rho^*)^{**} = \rho^*$. Restricting again to the hyperplane $t = \beta$ we see that $\mathcal{T}_\alpha^3\varphi = \mathcal{T}_\alpha\varphi$, which proves (2).

Next we prove (3), and begin by showing that $\mathcal{T}_\alpha$ is not surjective. If φ is in the image of $\mathcal{T}_\alpha$, then the above discussion implies that the corresponding ρ must be a norm on $\mathbb{R}^{n+1}$. In particular, ρ must be comparable to the Euclidean norm, i.e. there exists a constant $C > 0$ such that

$$\rho(x,t) \le C\left|(x,t)\right| = C\sqrt{|x|^2 + t^2}.$$

Therefore

$$\varphi(x) = \sqrt{\beta}\cdot\rho\left(x,\sqrt{\beta}\right) - \beta \le C\sqrt{\beta}\sqrt{|x|^2+\beta} \le C\left(\sqrt{\beta}\,|x| + \beta\right),$$

and we see that every function φ in the image of $\mathcal{T}_\alpha$ must grow at most linearly. In particular, the function $\varphi(x) = |x|^2$ is not in the image of $\mathcal{T}_\alpha$, so $\mathcal{T}_\alpha$ is not surjective.

Finally, we will show that $\mathcal{T}_\alpha$ is also not injective. Take any $\varphi \in \mathrm{Cvx}_0(\mathbb{R}^n)$ which is not in the image of $\mathcal{T}_\alpha$. Then $\mathcal{T}_\alpha\left(\mathcal{T}_\alpha^2\varphi\right) = \mathcal{T}_\alpha^3\varphi = \mathcal{T}_\alpha\varphi$, even though $\mathcal{T}_\alpha^2\varphi \ne \varphi$. This shows that $\mathcal{T}_\alpha$ is not injective, and the proof is complete.

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