

α -CONCAVE FUNCTIONS AND A FUNCTIONAL EXTENSION OF MIXED VOLUMES

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ABSTRACT. Mixed volumes, which are the polarization of volume with respect to the Minkowski addition, are fundamental objects in convexity. In this note we announce the construction of mixed integrals, which are functional analogs of mixed volumes. We build a natural addition operation \oplus on the class of quasi-concave functions, such that every class of α -concave functions is closed under \oplus . We then define the mixed integrals, which are the polarization of the integral with respect to \oplus .

We proceed to discuss the extension of various classic inequalities to the functional setting. For general quasi-concave functions, this is done by restating those results in the language of rearrangement inequalities. Restricting ourselves to α -concave functions, we state a generalization of the Alexandrov inequalities in their more familiar form.

1. α -CONCAVE FUNCTIONS

Let us begin by introducing our main objects of study:

Definition 1. Fix $-\infty \leq \alpha \leq \infty$. We say that a function $f : \mathbb{R}^n \rightarrow [0, \infty)$ is α -concave if f is supported on some convex set Ω , and for every $x, y \in \Omega$ and $0 \leq \lambda \leq 1$ we have

$$f(\lambda x + (1 - \lambda)y) \geq [\lambda f(x)^\alpha + (1 - \lambda) f(y)^\alpha]^{\frac{1}{\alpha}}.$$

For simplicity, we will always assume f is upper semicontinuous, $\max_{x \in \mathbb{R}^n} f(x) = 1$, and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$. The class of all such α -concave functions will be denoted by $C_\alpha(\mathbb{R}^n)$.

In the above definition, we follow the convention set by Brascamp and Lieb ([6]), but the notion of α -concavity may be traced back to Avriel ([1]) and Borell ([4], [5]). Discussions of α -concave functions from a geometric point of view may be found, e.g., in [2] and [9].

In the cases $\alpha = -\infty, 0, \infty$ we understand Definition 1 in the limit sense. For example, $f \in C_\infty(\mathbb{R}^n)$ if f is supported on some convex set Ω , and

$$f(\lambda x + (1 - \lambda)y) \geq \max\{f(x), f(y)\}$$

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for all $x, y \in \Omega$ and $0 \leq \lambda \leq 1$. Of course, this just means that f is constant on Ω . In other words, we have a natural correspondence between $C_\infty(\mathbb{R}^n)$ and the class \mathcal{K}_c^n of compact, convex sets in \mathbb{R}^n : every function $f \in C_\infty(\mathbb{R}^n)$ is of the form

$$f(x) = \mathbf{1}_K(x) = \begin{cases} 1 & x \in K \\ 0 & \text{otherwise,} \end{cases}$$

for some $K \in \mathcal{K}_c^n$.

Notice that if $\alpha_1 < \alpha_2$ then $C_{\alpha_1}(\mathbb{R}^n) \supset C_{\alpha_2}(\mathbb{R}^n)$ (see [6]). Therefore, we can view the class $C_\alpha(\mathbb{R}^n)$ for $\alpha < \infty$ as an extension of the class \mathcal{K}_c^n of convex sets. Our main goal in this note is to extend the geometric notion of mixed volumes from \mathcal{K}_c^n to the different classes of α -concave functions.

For convenience, we will restrict ourselves to the case $-\infty \leq \alpha \leq 0$. For $-\infty < \alpha < 0$, it is easy to see that f is α -concave if and only if f^α is a convex function on \mathbb{R}^n . The cases $\alpha = 0, -\infty$ are important, and deserve a special name:

Definition 2. (1) A 0-concave function is called log-concave. These are the functions $f : \mathbb{R}^n \rightarrow [0, \infty)$ such that

$$f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$. We will usually write $\text{LC}(\mathbb{R}^n)$ instead of $C_0(\mathbb{R}^n)$.

(2) A $(-\infty)$ -concave function is called quasi-concave. These are the functions $f : \mathbb{R}^n \rightarrow [0, \infty)$ such that

$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$$

for all $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$. We will usually write $\text{QC}(\mathbb{R}^n)$ instead of $C_{-\infty}(\mathbb{R}^n)$.

We will now see that if $-\infty < \alpha \leq 0$, there is a natural correspondence between $C_\alpha(\mathbb{R}^n)$ and convex functions on \mathbb{R}^n . Since we only care about negative values of α , it will sometimes be convenient to use the parameter $\beta = -\frac{1}{\alpha}$. The following definition appeared in [9]:

Definition 3. The convex base of a function $f \in C_\alpha(\mathbb{R}^n)$ is

$$\text{base}_\alpha(f) = \frac{1 - f^\alpha}{\alpha}.$$

Put differently, $\varphi = \text{base}_\alpha(f)$ is the unique convex function such that

$$f = \left(1 + \frac{\varphi}{\beta}\right)^{-\beta}.$$

In the limiting case $\alpha = 0$ we define $\text{base}_0(f) = -\log f$.

By our assumptions on f , the function $\varphi = \text{base}_\alpha(f)$ is convex, lower semicontinuous, with $\min \varphi = 0$ and $\varphi(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. We will denote this class of convex functions by $\text{Cvx}(\mathbb{R}^n)$ (this is not an entirely standard notation), and notice that the map base_α is a bijection between $C_\alpha(\mathbb{R}^n)$ and $\text{Cvx}(\mathbb{R}^n)$. It follows immediately, for example, that every function $f \in C_\alpha(\mathbb{R}^n)$ is continuous on its support, because the same is true for convex functions.

In the case $\alpha = -\infty$, we have no such correspondence. It is therefore not surprising that it is possible to construct quasi-concave functions which are not continuous on their support. Indeed, fix convex sets $K_1 \subset K_2$ and define $f = \mathbf{1}_{K_1} + \mathbf{1}_{K_2}$.

Remember that if $f \in C_\alpha(\mathbb{R}^n)$, then $f \in C_{\alpha'}(\mathbb{R}^n)$ for every $\alpha' < \alpha$. However, in general we have $\text{base}_\alpha(f) \neq \text{base}_{\alpha'}(f)$, so the base depends on the class we choose to work in, and not only on our function f . However, in the specific case $f = \mathbf{1}_K$ for some convex set K , we have

$$\text{base}_\alpha(f) = \mathbf{1}_K^\infty = \begin{cases} 0 & x \in K \\ \infty & \text{otherwise} \end{cases}$$

for every value of α .

On $\text{Cvx}(\mathbb{R}^n)$ there is a natural addition operation, known as inf-convolution:

Definition 4. For $\varphi, \psi \in \text{Cvx}(\mathbb{R}^n)$ we define their inf-convolution to be

$$(\varphi \square \psi)(x) = \inf_{y+z=x} [\varphi(y) + \psi(z)].$$

Similarly, if $\varphi \in \text{Cvx}(\mathbb{R}^n)$ and $\lambda > 0$ we will define

$$(\lambda \cdot \varphi)(x) = \lambda \varphi\left(\frac{x}{\lambda}\right).$$

The definition of $\lambda \cdot \varphi$ was chosen to have $2 \cdot \varphi = \varphi \square \varphi$, as one easily verifies. It is also easy to see that we have commutativity, associativity and distributivity.

We will not explain the exact sense in which these operations are natural, and instead refer the reader to the first section of [9]. We will note, however, that Definition 4 extends the classical operations on convex bodies: If $K_1, K_2 \in \mathcal{K}_c^n$ and $\lambda > 0$ then

$$(\lambda \cdot \mathbf{1}_{K_1}^\infty) \square \mathbf{1}_{K_2}^\infty = \mathbf{1}_{\lambda K_1 + K_2}^\infty.$$

Here $+$ is the Minkowski sum of convex bodies, defined by

$$K_1 + K_2 = \{x + y : x \in K_1, y \in K_2\},$$

and λK is defined by

$$\lambda K = \{\lambda x : x \in K\}.$$

We will now define addition on $C_\alpha(\mathbb{R}^n)$, using the established correspondence between $C_\alpha(\mathbb{R}^n)$ and $\text{Cvx}(\mathbb{R}^n)$:

Definition 5. Fix $-\infty < \alpha \leq 0$. Then:

- (1) For $f, g \in C_\alpha(\mathbb{R}^n)$ we define their α -sum $f \star_\alpha g$ by the relation

$$\text{base}_\alpha(f \star_\alpha g) = (\text{base}_\alpha f) \square (\text{base}_\alpha g).$$

- (2) For $f \in C_\alpha(\mathbb{R}^n)$ and $\lambda > 0$ we define $\lambda \cdot_\alpha f$ via the relation

$$\text{base}_\alpha(\lambda \cdot_\alpha f) = \lambda \cdot \text{base}_\alpha f.$$

Again, the definition of α -sum depends on α , and not only on f and g : If $\alpha' < \alpha$ and $f, g \in C_\alpha(\mathbb{R}^n)$, then in general we have $f \star_\alpha g \neq f \star_{\alpha'} g$. However, for indicators of convex sets we have

$$\mathbf{1}_{K_1} \star_\alpha \mathbf{1}_{K_2} = \mathbf{1}_{K_1 + K_2}$$

for all α .

The definition of α -sum may be written down explicitly, without referring to the convex bases. For $-\infty < \alpha < 0$ we have

$$(1.1) \quad (f \star_\alpha g)(x) = \sup_{y+z=x} (f(y)^\alpha + g(z)^\alpha - 1)^{\frac{1}{\alpha}},$$

and for $\alpha = 0$ we get the limiting case

$$(f \star_0 g)(x) = \sup_{y+z=x} f(y)g(z).$$

The operation \star_0 on $\text{LC}(\mathbb{R}^n)$ is known as Asplund-sum, or sup-convolution (see, e.g., [7]).

For $\alpha = -\infty$ we cannot define $f \star g$ using the same approach as Definition 5, because we do not have the notion of a base for quasi-concave functions. However, we may use equation 1.1, and the fact that for every $0 < u, v \leq 1$ we have

$$\lim_{\alpha \rightarrow -\infty} [u^\alpha + v^\alpha - 1]^{\frac{1}{\alpha}} = \min\{u, v\}.$$

This, and a similar consideration for \cdot_α , leads us to define:

Definition 6. (1) For $f, g \in \text{QC}(\mathbb{R}^n)$ we define their quasi-sum $f \oplus g$ by

$$(f \oplus g)(x) = \sup_{y+z=x} \min\{f(y), g(z)\}.$$

(2) For $f \in \text{QC}(\mathbb{R}^n)$ and $\lambda > 0$ we define $\lambda \odot f$ by

$$(\lambda \odot f)(x) = f\left(\frac{x}{\lambda}\right).$$

For $\lambda = 0$, we explicitly define

$$(0 \odot f)(x) = \mathbf{1}_{\{0\}}(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

This definition ensures that $f \oplus (0 \odot g) = f$ for every $f, g \in \text{QC}(\mathbb{R}^n)$. We use the notations \oplus and \odot instead of $\star_{-\infty}$ and $\cdot_{-\infty}$ because these operations will play a fundamental role in the rest of this paper.

So far we discussed properties of α -concave functions which made sense for every value of α . We now want to state a few results that are only true for quasi-concave functions and quasi-sums. We will need the following definition:

Definition 7. For a function $f : \mathbb{R}^n \rightarrow [0, 1]$ and $0 < t \leq 1$ we define

$$K_t(f) = \{x : f(x) \geq t\}$$

to be the upper level sets of f .

We now have the following result, which will play an important role in this note:

Theorem 8. (1) Fix $f : \mathbb{R}^n \rightarrow [0, 1]$. Then $f \in \text{QC}(\mathbb{R}^n)$ if and only if $K_t(f)$ are compact, convex sets for all $0 < t \leq 1$.

(2) For every $f, g \in \text{QC}(\mathbb{R}^n)$, $\lambda > 0$ and $0 < t \leq 1$ we have

$$K_t((\lambda \odot f) \oplus g) = \lambda K_t(f) + K_t(g).$$

The sum \oplus has another important property we would now like to discuss. Remember that if $f, g \in C_\alpha(\mathbb{R}^n)$, then $f, g \in C_{\alpha'}(\mathbb{R}^n)$ for all $\alpha' < \alpha$, so we may look at the function $h = f \star_{\alpha'} g \in C_{\alpha'}(\mathbb{R}^n)$. Generally, the function h does not have to be in $C_\alpha(\mathbb{R}^n)$, even though f and g are. Let us consider an example: choose $f(x) = g(x) = e^{-|x|} \in \text{LC}(\mathbb{R}^n)$, and choose $\alpha' = -1$. In this case we have

$$\text{base}_{(-1)} f = \text{base}_{(-1)} g = e^{|x|} - 1,$$

and since $h = f \star_{\alpha'} g = 2 \cdot_{\alpha'} f$ we have

$$\text{base}_{(-1)} h = 2 \cdot \left(e^{\frac{|x|}{2}} - 1 \right).$$

Therefore

$$h(x) = \frac{1}{2e^{\frac{|x|}{2}} - 1},$$

and it is easy to check that $h \notin \text{LC}(\mathbb{R}^n)$. Of course, we must have $h \in C_{-1}(\mathbb{R}^n)$.

It turns out that such a situation cannot happen when $\alpha' = -\infty$:

Theorem 9. *If $f, g \in C_\alpha(\mathbb{R}^n)$ for some $-\infty \leq \alpha \leq 0$, so does $f \oplus g$.*

The proofs of the last two results will appear in [8]. Notice that by this theorem we have two different addition operations on $C_\alpha(\mathbb{R}^n)$. One is \star_α , and the second is the “universal” \oplus .

2. MIXED INTEGRALS

Recall the following theorem by Minkowski (see, e.g., [10] for a proof):

Theorem (Minkowski). *Fix $K_1, K_2, \dots, K_m \in \mathcal{K}_c^n$. Then $F : (\mathbb{R}^+)^m \rightarrow [0, \infty)$ defined by*

$$F(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) = \text{Vol}(\varepsilon_1 K_1 + \varepsilon_2 K_2 + \dots + \varepsilon_m K_m)$$

is a homogenous polynomial of degree n , with non-negative coefficients.

The coefficients of this polynomial are called mixed volumes. To be more exact, we have a function

$$V : (\mathcal{K}_c^n)^n \rightarrow [0, \infty)$$

which is multilinear (with respect to the Minkowski sum), symmetric (i.e., invariant to a permutation of its arguments), and which satisfies $V(K, K, \dots, K) = \text{Vol}(K)$. From these properties it is easy to deduce that

$$F(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) = \sum_{i_1, i_2, \dots, i_n=1}^m \varepsilon_{i_1} \varepsilon_{i_2} \dots \varepsilon_{i_n} \cdot V(K_{i_1}, K_{i_2}, \dots, K_{i_n}).$$

The number $V(K_1, K_2, \dots, K_n)$ is called the mixed volume of the K_1, K_2, \dots, K_n .

As we stated before, our goal is to prove a functional extension of Minkowski’s theorem, and to define a functional extension of mixed volumes. We will state our results on $\text{QC}(\mathbb{R}^n)$, since this is the largest class of functions we consider, so all the results will be true for every class $C_\alpha(\mathbb{R}^n)$ (and, in particular, on $\text{LC}(\mathbb{R}^n)$). Of course, in order to formulate and prove such a theorem, we need to decide what are the functional analogs of volume, and of Minkowski sum.

For volume, if we want our theorem to be a true extension of Minkowski’s, we need a functional Φ on $\text{QC}(\mathbb{R}^n)$ such that $\Phi(\mathbf{1}_K) = \text{Vol}(K)$. A natural candidate is the Lebesgue integral,

$$\Phi(f) = \int_{\mathbb{R}^n} f(x) dx.$$

For the extension of addition, it turns out that the best possibility is the quasi-sum \oplus . In fact, we have the following theorem:

Theorem 10. *Fix $f_1, f_2, \dots, f_m \in \text{QC}(\mathbb{R}^n)$. Then $F : (\mathbb{R}^+)^m \rightarrow [0, \infty]$ defined by*

$$F(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) = \int [(\varepsilon_1 \odot f_1) \oplus (\varepsilon_2 \odot f_2) \oplus \dots \oplus (\varepsilon_m \odot f_m)]$$

is a homogenous polynomial of degree n , with non-negative coefficients.

The proof will appear in [8]. In complete analogy with the case of convex bodies, this theorem is equivalent to the existence of a function

$$V : \text{LC}(\mathbb{R}^n)^n \rightarrow [0, \infty]$$

which is symmetric, multilinear (with respect to \oplus , of course) and satisfies

$$V(f, f, \dots, f) = \int_{\mathbb{R}^n} f(x) dx.$$

We will call the number $V(f_1, f_2, \dots, f_n)$ the mixed integral of f_1, f_2, \dots, f_n . In other words, the mixed integral is the polarization of the integral $\int f$. We have the following representation formula for the mixed integrals:

Proposition 11. *Fix $f_1, f_2, \dots, f_n \in \text{QC}(\mathbb{R}^n)$. Then*

$$V(f_1, f_2, \dots, f_n) = \int_0^1 V(K_t(f_1), K_t(f_2), \dots, K_t(f_n)) dt.$$

Mixed integrals share many important properties with the classical mixed volumes. We will mention a few in the following theorem:

Theorem 12. (1) *For $K_1, K_2, \dots, K_n \in \mathcal{K}_c^n$ we have*

$$V(K_1, K_2, \dots, K_n) = V(\mathbf{1}_{K_1}, \mathbf{1}_{K_2}, \dots, \mathbf{1}_{K_n}).$$

(2) *For every $f_1, f_2, \dots, f_n \in \text{QC}(\mathbb{R}^n)$ we have $V(f_1, f_2, \dots, f_n) \geq 0$. More generally, if we also have $g_1, g_2, \dots, g_n \in \text{QC}(\mathbb{R}^n)$ such that $f_i \geq g_i$ for all i , then*

$$V(f_1, f_2, \dots, f_n) \geq V(g_1, g_2, \dots, g_n).$$

(3) *V is rotation and translation invariant. Also, if we define*

$$(uf)(x) = f(u^{-1}x)$$

for $f \in \text{QC}(\mathbb{R}^n)$ and $u \in \text{GL}(n)$, then

$$V(uf_1, uf_2, \dots, uf_n) = |\det u| \cdot V(f_1, f_2, \dots, f_n).$$

(4) *Let $f_1, f_2, \dots, f_n \in \text{QC}(\mathbb{R}^n)$ and denote by K_i the support of f_i . Then $V(f_1, f_2, \dots, f_n) = 0$ if and only if $V(K_1, K_2, \dots, K_n) = 0$.*

(5) *Fix an integer $1 \leq m \leq n$ and functions $g_{m+1}, \dots, g_n \in \text{QC}(\mathbb{R}^n)$. Then the functional*

$$\Phi(f) = V(f[m], g_{m+1}, \dots, g_n)$$

satisfies a valuation type property: if $f_1, f_2 \in \text{QC}(\mathbb{R}^n)$ and $f_1 \vee f_2 = \max(f_1, f_2) \in \text{QC}(\mathbb{R}^n)$ as well, then

$$\Phi(f_1 \vee f_2) + \Phi(f_1 \wedge f_2) = \Phi(f_1) + \Phi(f_2).$$

Here $f_1 \wedge f_2$ is an alternative notation for $\min\{f_1, f_2\}$.

These properties are deduced by using Proposition 11 and the corresponding properties for mixed volumes. We will prove claim 4, and leave the others to the reader:

Proof. Denote $V = V(K_1, K_2, \dots, K_n)$, and define $f : (0, 1] \rightarrow \mathbb{R}$ by

$$f(t) = V(K_t(f_1), K_t(f_2), \dots, K_t(f_n)).$$

notice that f is non-negative and non-increasing, by monotonicity of mixed volumes. Since

$$K_i = \overline{\bigcup_{t>0} K_t(f_i)},$$

(where the bar denotes the topological closure), continuity of mixed volumes gives $\lim_{t \rightarrow 0^+} f(t) = V$.

Therefore, if $V = 0$ then $f \equiv 0$, so $V(f_1, f_2, \dots, f_n) = 0$. If, on the other hand, $V > 0$, then $f(t) > \frac{V}{2}$ for all t smaller than some t_0 , so

$$V(f_1, f_2, \dots, f_n) = \int_0^1 f(t) dt \geq \int_0^{t_0} f(t) dt = \frac{t_0 V}{2} > 0.$$

□

A particularly interesting example of mixed volumes is quermassintegrals. For $K \in \mathcal{K}_c^n$, we define the k -th quermassintegral to be

$$W_k(K) = V(\underbrace{K, K, \dots, K}_{n-k \text{ times}}, \underbrace{D, D, \dots, D}_{k \text{ times}}),$$

where D is the Euclidean unit ball. Similarly, for $f \in \text{QC}(\mathbb{R}^n)$ we will define

$$W_k(f) = V(\underbrace{f, f, \dots, f}_{n-k \text{ times}}, \underbrace{\mathbf{1}_D, \mathbf{1}_D, \dots, \mathbf{1}_D}_{k \text{ times}}).$$

By checking the definitions, we see that if $f \in C_\alpha(\mathbb{R}^n)$ and K is a convex set, then

$$[f \star_{\alpha'} (\varepsilon \cdot_{\alpha'} \mathbf{1}_K)](x) = \sup_{y \in \varepsilon K} f(x - y)$$

for every $\alpha' \leq \alpha$. In particular, the left hand side is independent of the exact value α' . Since for $\alpha' = -\infty$ we obtain a polynomial in ε , we must obtain the same polynomial for every value of α' . Therefore, as a direct corollary of Theorem 10 we obtain the following statement, which was independently obtained by Bobkov, Colesanti and Fragalà (see [3]):

Proposition 13. *Fix $-\infty \leq \alpha' \leq \alpha \leq 0$. For $f \in C_\alpha(\mathbb{R}^n)$ and $\varepsilon > 0$, define*

$$f_\varepsilon(x) = [f \star_{\alpha'} (\varepsilon \cdot_{\alpha'} \mathbf{1}_D)](x) = \sup_{|y| \leq \varepsilon} f(x + y).$$

Then we have

$$\int f_\varepsilon = \sum_{i=0}^n \binom{n}{i} W_i(f) \varepsilon^i.$$

As stated, this result was also discovered by Bobkov, Colesanti and Fragalà. Their paper continues to prove several properties of the quermassintegrals, such as Prékopa-Leindler inequalities and a Cauchy-Kubota integral formula. We will not pursue these points in this note. Let us stress that Proposition 13 only works for quermassintegrals, where the different notions of sum happen to coincide. For general mixed integrals, it is impossible to get polynomiality for the operation \star_α unless $\alpha = -\infty$.

3. INEQUALITIES

Now that we have a functional version of Minkowski’s theorem, we would like to prove inequalities between different mixed integrals. Let us use the isoperimetric inequality as a test case. The classical isoperimetric inequality, arguably the most famous inequality in geometry, claims that for every (say convex) body $K \in \mathcal{K}_c^n$ we have

$$S(K) \geq n \cdot \text{Vol}(D)^{\frac{1}{n}} \cdot \text{Vol}(K)^{\frac{n-1}{n}}.$$

Here $S(K)$ is the surface area of K , defined by

$$S(K) = \lim_{\varepsilon \rightarrow 0^+} \frac{\text{Vol}(K + \varepsilon D) - \text{Vol}(K)}{\varepsilon} = n \cdot W_1(K).$$

We would like to generalize this result to the functional setting. The naive approach would be to try and bound $S(f) := nW_1(f)$ from below using $\int f$. Unfortunately, this is impossible to do for general quasi-concave functions. In fact, it is possible to construct a sequence of functions $f_k \in \text{QC}(\mathbb{R}^n)$ such that $\int f_k = 1$ but $S(f_k) \rightarrow 0$ as $k \rightarrow \infty$. We will present a concrete example in [8].

Thus we will use a different approach, and prove an extension of the isoperimetric inequality by recasting it as a rearrangement inequality. In order to explain this idea, consider the following definition:

Definition 14. (1) For a compact $K \in \mathcal{K}_c^n$, define

$$K^* = \left(\frac{\text{Vol}(K)}{\text{Vol}(D)} \right)^{\frac{1}{n}} D.$$

In other words, K^* is the Euclidean ball with the same volume as K .

(2) For $f \in \text{QC}(\mathbb{R}^n)$, define its symmetric decreasing rearrangement f^* using the relation

$$\overline{K}_t(f^*) = \overline{K}_t(f)^*.$$

It is easy to see that this definition really defines a unique function $f^* \in \text{QC}(\mathbb{R}^n)$, which is rotation invariant.

Now, the isoperimetric inequality may be restated as $S(K) \geq S(K^*)$ for $K \in \mathcal{K}_c^n$. In this formulation, the functional extension turns out to be true:

Proposition 15. *If $f \in \text{QC}(\mathbb{R}^n)$, then $S(f) \geq S(f^*)$, with equality if and only if f is rotation invariant.*

This inequality is indeed an extension of the isoperimetric inequality, as can be seen by choosing $f = \mathbf{1}_K$. It can also be useful for general quasi-concave functions, because it reduces an n -dimensional problem to a 1-dimensional one – the function f^* is rotation invariant, and hence essentially “one dimensional”. However, we stress again that in general, this inequality does not yield a lower bound for $S(f)$ in terms of $\int f$, as such a bound is impossible.

Many other inequalities can be extended using similar formulations. For example, the Brunn-Minkowski inequality states that for every (say convex) sets $A, B \in \mathcal{K}_c^n$ we have

$$\text{Vol}(A + B)^{\frac{1}{n}} \geq \text{Vol}(A)^{\frac{1}{n}} + \text{Vol}(B)^{\frac{1}{n}}.$$

Again, in general, it is impossible to bound $\int (f \oplus g)$ from below using $\int f$ and $\int g$. However, the Brunn-Minkowski inequality may be written as $(A + B)^* \supseteq A^* + B^*$, and in this representation it generalizes well:

Theorem 16. *For every $f, g \in \text{QC}(\mathbb{R}^n)$ we have $(f \oplus g)^* \geq f^* \oplus g^*$.*

In [8] we will prove the above two results, as well as extensions of the generalized Brunn-Minkowski inequalities for mixed volumes and the Alexandrov-Fenchel inequality. We will not describe these results here, since they require the notion of a “generalized rearrangement”. Instead, let us mention one elegant corollary of the Alexandrov-Fenchel inequality. Remember that for convex bodies we have the inequality

$$V(K_1, K_2, \dots, K_n) \geq \left[\prod_{i=1}^n \text{Vol}(K_i) \right]^{\frac{1}{n}}.$$

The functional analog of this result is the following inequality:

Theorem 17. *For all functions $f_1, \dots, f_n \in \text{QC}(\mathbb{R}^n)$ we have*

$$V(f_1, f_2, \dots, f_n) \geq V(f_1^*, f_2^*, \dots, f_n^*).$$

Notice that Theorem 17 is a generalization of the isoperimetric inequality of Proposition 15, and it can also be used to deduce an Urysohn type inequality, bounding $W_{n-1}(f)$ using $W_{n-1}(f^*)$.

If one is willing to restrict oneself to some class of α -concave functions, then it is suddenly possible to prove inequalities between mixed integrals in a more familiar form. In order to state the result, let us define for every $-\infty < \alpha \leq 0$ a function $g_\alpha \in C_\alpha(\mathbb{R}^n)$ by

$$g_\alpha(x) = \left(1 + \frac{|x|}{\beta} \right)^{-\beta},$$

where, as usual $\beta = -\frac{1}{\alpha}$. Put differently, we choose g_α to satisfy $\text{base}_\alpha(g_\alpha) = |x|$ (for $\alpha = 0$, we obtain $g_0(x) = e^{-|x|}$). By abuse of notation, we will also think of g_α as the function from $[0, \infty)$ to $[0, \infty)$ defined by

$$g_\alpha(r) = \left(1 + \frac{r}{\beta} \right)^{-\beta},$$

so $g_\alpha(x) = g_\alpha(|x|)$. We are now ready to state:

Theorem 18. *Fix a function $f \in C_\alpha(\mathbb{R}^n)$ and integers k and m such that $0 \leq k < m < n$. Then we have*

$$\left(\frac{W_k(f)}{W_k(g_\alpha)} \right)^{\frac{1}{n-k}} \leq \left(\frac{W_m(f)}{W_m(g_\alpha)} \right)^{\frac{1}{n-m}},$$

assuming $W_k(g_\alpha) < \infty$. Equality occurs if and only if $f = \lambda \odot g_\alpha$ for some $\lambda \geq 0$.

Of course, if $W_k(g_\alpha) = \infty$ the theorem is either trivial (if $W_k(f) < \infty$) or meaningless (if $W_k(f) = \infty$). The condition $W_k(g_\alpha) < \infty$ is equivalent to $k > n + \frac{1}{\alpha}$, and implies that all the other quantities in the theorem are finite as well. Notice that since $k < m < n$ are all integers, we have $k \leq n - 2$. Hence we need to choose $\alpha > -\frac{1}{2}$ for the theorem to have any content.

In [8], a proof will be given for the case $\alpha = 0$, where the condition $W_k(g_\alpha) < \infty$ is true for all k . A key ingredient in the proof is a bound on the growth of moments of log-concave functions. The general proof is similar, and depends on the following lemma:

Lemma 19. *Let $f : [0, \infty) \rightarrow [0, \infty)$ be an α -concave function such that $f(0) = 1$. Then for every $0 \leq k < m < -\frac{1}{\alpha} - 1$ we have*

$$\left(\frac{\int_0^\infty r^m f(r) dr}{\int_0^\infty r^m g_\alpha(r) dr} \right)^{\frac{1}{m+1}} \leq \left(\frac{\int_0^\infty r^k f(r) dr}{\int_0^\infty r^k g_\alpha(r) dr} \right)^{\frac{1}{k+1}},$$

with equality if and only if

$$f(r) = (\lambda \odot g_\alpha)(r) = \left(1 + \frac{r}{\lambda\beta} \right)^{-\beta}$$

for some λ .

Again, the condition $m < -\frac{1}{\alpha} - 1$ simply ensures that all of the integrals are finite. Under this condition, the lemma follows from Lemma 4.2 of [2] by taking $Q = g_\alpha$ (In [2] the equality condition is not explicitly stated, but it can be deduced by carefully inspecting the proof).

We will conclude by sketching the proof of Theorem 18. Some parts of the proof, which are identical to the log-concave case, will be glossed over and explained fully in [8].

Proof. First, we reduce the general case to the rotation invariant case by replacing f with some generalized rearrangement f^{W_k} . The definition of f^{W_k} and its necessary properties will appear in [8].

So, assume without loss of generality that f is rotation invariant. By abuse of notation we will write $f(x) = f(|x|)$. A direct computation (to also appear in [8]) gives

$$W_i(f) = (n - i) \cdot \text{Vol}(D) \cdot \int_0^\infty r^{n-i-1} f(r) dr,$$

so

$$\left(\frac{W_k(f)}{W_k(g_\alpha)} \right)^{\frac{1}{n-k}} = \left(\frac{\int_0^\infty r^{n-k-1} f(r) dr}{\int_0^\infty r^{n-k-1} g_\alpha(r) dr} \right)^{\frac{1}{n-k}},$$

and similarly for m . Remember that we assumed $W_k(g_\alpha) < \infty$, which is the same as

$$\int_0^\infty r^{n-k-1} \left(1 + \frac{r}{\beta} \right)^{-\beta} dr < \infty.$$

This implies that $n - k - 1 - \beta < -1$, or $k > n + \frac{1}{\alpha}$, like we claimed. In particular we have

$$0 \leq n - m - 1 < n - k - 1 < -\frac{1}{\alpha} - 1,$$

so we can use Lemma 19 and conclude that

$$\left(\frac{\int_0^\infty r^{n-k-1} f(r) dr}{\int_0^\infty r^{n-k-1} g_\alpha(r) dr} \right)^{\frac{1}{n-k}} \leq \left(\frac{\int_0^\infty r^{n-m-1} f(r) dr}{\int_0^\infty r^{n-m-1} g_\alpha(r) dr} \right)^{\frac{1}{n-m}}.$$

This is the same as

$$\left(\frac{W_k(f)}{W_k(g_\alpha)} \right)^{\frac{1}{n-k}} \leq \left(\frac{W_m(f)}{W_m(g_\alpha)} \right)^{\frac{1}{n-m}},$$

which is what we wanted.

The equality case will follow from the equality case of Lemma 19, but we will not give the details here. \square

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