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Characterizing addition of convex sets by polynomiality of volume and by the homothety operation

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We study addition operations between convex sets. We show that, under a short list of natural assumptions, one has polynomiality of volume only for the Minkowski addition.

We also give two other characterization theorems. For the first theorem we define the induced homothety of an addition operation, and characterize additions by this homothety. The second theorem characterizes all additions which satisfy a short list of natural conditions.

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1. Introduction

One of the most fundamental theorems in convex geometry is Minkowski's theorem on polynomiality of volume. To explain this theorem, let us denote by $\mathcal{K}_0^n$ the class of all closed convex sets K in $\mathbb{R}^n$ such that $0 \in K$ (this last condition is not important right now, but will be used extensively later). For $A, B \in \mathcal{K}_0^n$, we define their Minkowski sum to be

$$A + B = \{a + b : \ a \in A, \ b \in B\}.$$

Similarly, if $A \in \mathcal{K}_0^n$ and $\lambda > 0$, we define the homothety λA as

$$\lambda A = \{\lambda a : \ a \in A\}.$$

Under these definitions we have all of the expected equalities, such as $\lambda A + \mu A = (\lambda + \mu) A$.

Finally, if $A \in \mathcal{K}_0^n$ we denote by $|A| \in [0, \infty]$ its (Lebesgue) volume.

We are now ready to state:

Theorem 1.1 (Minkowski). *Fix $A_1, A_2, \dots, A_m \in \mathcal{K}_0^n$. Then the function $F : (\mathbb{R}^+)^m \to [0,\infty]$, defined by*

$$F(\lambda_1, \lambda_2, \dots, \lambda_m) = |\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m|,$$

is a homogenous polynomial of degree n, with positive coefficients.

The coefficients of the polynomial F are known as mixed volumes (after a proper normalization). Since our sets $A \in \mathcal{K}_0^n$ may have infinite volume, some of these coefficients may be $+\infty$. This does not cause a problem, as long as we adopt the convention that $0 \cdot \infty = 0$. For a proof of Minkowski's theorem, as well as more information about mixed volumes, see chapter 5 of [12].

There are many other interesting addition operations on convex sets other than the Minkowski addition, and we will give several such examples in the next section. While these operations are definitely important, and some of them even yielded an entire theory (such as the "L_p-Brunn Minkowski theory"), they do not satisfy a theorem analogous to Minkowski's. Hence, a natural question presents itself: Does polynomiality of volume characterize the Minkowski addition uniquely? In other words, assume we are given some addition operation $\oplus$, and the corresponding homothety operation $\odot$ (this notions will be made exact in the next section). If

$$F(\lambda_1, \lambda_2, \dots, \lambda_m) = |(\lambda_1 \odot A_1) \oplus (\lambda_2 \odot A_2) \oplus \dots \oplus (\lambda_m \odot A_m)|$$

is always a polynomial, does it follow that $\oplus$ must be the Minkowski addition?

The situation becomes even more complicated when one replaces sets with functions. In recent years, there was a surge of interesting results in convex geometry and geometric analysis, which were obtained by extending the class of convex sets to a larger class of functions, satisfying a convexity assumption. The most standard choice of such a class seems to be the class of log-concave functions: these are the functions $f : \mathbb{R}^n \to [0,\infty)$ such that $(-\log f)$ is a convex function. Let us denote by $\mathrm{LC}_0(\mathbb{R}^n)$ the class of all log-concave functions which are upper semi-continuous and which satisfy

$$\max f = f(0) = 1.$$

On the class $\mathrm{LC}_0(\mathbb{R}^n)$ there are even more possible addition operations then on the class $\mathcal{K}_0^n$. Again, for the vast majority of these sums there will be no analog of Minkowski's theorem. Recently, we introduced a new sum $\oplus$ on $\mathrm{LC}_0(\mathbb{R}^n)$ (and in fact, on the larger class of quasi-concave functions), for which we have polynomiality: the integral

$$\int ((\lambda_1 \odot f_1) \oplus (\lambda_2 \odot f_2) \oplus \dots \oplus (\lambda_m \odot f_m))$$

is a polynomial in $\lambda_1, \lambda_2, \dots, \lambda_m$ (see [9],[10]). Of course, the same question now presents itself: Does this property characterizes $\oplus$ uniquely?

In this paper we will answer this question in the affirmative for the classical case of convex sets. We will show that under some minimal conditions, natural for the notion of addition, the only addition which satisfies Minkowski's theorem is the Minkowski sum. We will also give two other such characterization theorems: the first will characterize the addition $\oplus$ by its induced homothety $\odot$, assuming very little on $\oplus$. The second will impose another list of natural conditions on $\oplus$, but assume nothing on $\odot$.

Let us note that a recent paper by Gardner, Hug and Weil also characterizes additions of convex sets ([4]). Their results are not directly comparable to ours, as the homothety operation they consider is the standard ("Minkowski") homothety λA, and not the homothety which corresponds to the addition operation. Additionally, we think that some of the conditions they impose are too restrictive. We will give more information about this in the next section.

In most of this paper, we will use $\mathcal{K}_0^n$ as our domain. In the last section, we will explain how our results can also be extended to additions on $\mathcal{K}_{0,c}^n$ - the class of compact sets inside $\mathcal{K}_0^n$. One can also modify our proofs to deal with the case of origin symmetric convex sets, compact or not, but as the proofs are so similar to the non-symmetric case we will not do so here.

2. Addition operations on convex bodies

In this note we are mainly interested in different "additions" on the class $\mathcal{K}_0^n$:

Definition 2.1. An *addition operation* on $\mathcal{K}_0^n$ is a map $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ such that

(1) $\oplus$ is associative and has a two-sided identity element (i.e. there exists $K \in \mathcal{K}_0^n$ such that $A \oplus K = K \oplus A = A$ for all $A \in \mathcal{K}_0^n$).
(2) $\oplus$ is monotone: If $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$, then $A_1 \oplus A_2 \subseteq B_1 \oplus B_2$.

Before we can state our main theorems, we need to describe a few important examples of such additions.

To begin, let us fix a Euclidean structure on $\mathbb{R}^n$. Remember that to any convex set $A \in \mathcal{K}_0^n$ we may associate a function $h_A : \mathbb{R}^n \to [0, \infty]$ called the support function of A, and defined by

$$h_A(\theta) = \sup_{x \in A} \langle x, \theta \rangle .$$

It is well-known that h_A is convex, positively homogeneous and lower semicontinuous. In the other direction, every convex, positively homogeneous and lower semicontinuous $f : \mathbb{R}^n \to [0, \infty]$ is of the form h_A for some $A \in \mathcal{K}_0^n$ (for proofs of these basic results, see e.g. section 13 of [11], or section 1.7.1 of [12] in the case that A is compact).

Hence we may define:

Definition 2.2. The p-addition of A and B is defined by the relation

$$h_{A+_pB}(\theta) = \left(h_A(\theta)^p + h_B(\theta)^p\right)^{\frac{1}{p}} .$$

p-sums of convex bodies were introduced by Firey ([3]) and studied extensively by Lutwak, who developed the so-called "L_p Brunn-Minkowski theory" (beginning with [7] and [8]). An organized discussion of these topics also appears in chapter 9 of [12]. We will only need the basic fact that $+_p$ is an addition on $\mathcal{K}_0^n$ (in particular, it preserves convexity) as long as $p \geq 1$.

Of particular interest is the case $p = 1$, where the body $A +_1 B$ is the closure of the Minkowski sum $A + B$:

$$A +_1 B = \overline{\{a + b : \ a \in A, \ b \in B\}}.$$

This closure is necessary because $A + B$ may not be closed, even if A and B are. Of course, if at least one of the sets A and B is compact, then $A +_1 B = A + B$.

As another example of an addition on convex bodies we define:

Definition 2.3. The ∞-addition of A and B is the closure of the convex hull of $A \cup B$:

$$A +_\infty B = \overline{\operatorname{conv}(A \cup B)}$$

Notice that

$$h_{A +_\infty B}(\theta) = \max\{h_A(\theta), h_B(\theta)\} = \lim_{p \to \infty} h_{A +_p B}(\theta),$$

which explains the name we use for this addition.

In order to define some more additions on $\mathcal{K}_0^n$ we need the definition of the polar body. Remember that if $A \in \mathcal{K}_0^n$ then

$$A^\circ = \{y \in \mathbb{R}^n : \ \langle x, y \rangle \leq 1 \text{ for all } x \in A\} \in \mathcal{K}_0^n.$$

Polarity is an order reversing involution on $\mathcal{K}_0^n$. Here "order reversing" means that if $A \subseteq B$ then $A^\circ \supseteq B^\circ$. "Involution" means, of course, that $A^{\circ\circ} = A$ (for a proof see section 1.6.1 of [12]). By these properties, it is easy to see that if $\oplus$ is an addition on $\mathcal{K}_0^n$, so is its "polar" $\boxplus$, defined by

$$A \boxplus B = (A^\circ \oplus B^\circ)^\circ.$$

In particular, we may define:

Definition 2.4. The $(-p)$-addition of A and B is defined by

$$A +_{(-p)} B = (A^\circ +_p B^\circ)^\circ.$$

By the above discussion, $+_{-p}$ is an addition operation on $\mathcal{K}_0^n$ for every $p \geq 1$. These additions were also defined by Firey ([2]) and developed by Lutwak ([8]). Again, a short discussion appears in chapter 9 of [12].

As a final addition operation we have

Definition 2.5. The $(-\infty)$-addition of A and B is just their intersection $A \cap B$.

The reason for the name is the relation

$$A +_{(-\infty)} B = (A^\circ +_\infty B^\circ)^\circ.$$

In [4] Gardner, Hug and Weil characterize additions on convex sets. Let us give an example of their results. We denote by $\mathcal{K}_s^n$ the class of compact, origin-symmetric, convex sets in $\mathbb{R}^n$. An operation $\oplus : \mathcal{K}_s^n \times \mathcal{K}_s^n \to \mathcal{K}_s^n$ is called projection covariant if for every subspace $E \subseteq \mathbb{R}^n$ we have

$$P_E\left(A \oplus B\right) = P_E A \oplus P_E B.$$

Here $P_E : \mathbb{R}^n \to E$ denotes the orthogonal projection. One of the main results of [4] then reads:

Theorem 2.1. *If $\oplus : \mathcal{K}_s^n \times \mathcal{K}_s^n \to \mathcal{K}_s^n$ is associative and projection covariant, then $\oplus = +_p$ for some $1 \le p \le \infty$ (up to three trivial exceptions).*

For us, the condition that $\oplus$ is projection covariant seems too restrictive, as this condition immediately implies that for every $\theta \in S^{n-1}$, $h_{A\oplus B}(\theta)$ depends only on $h_A(\theta)$ and $h_B(\theta)$. Hence, for example, it automatically excludes all p-additions for negative values of p (these additions are characterized in [4] by independent theorems, assuming section covariance). Instead, we would like to follow a different path, and characterize additions by other properties, which we feel are weaker and more natural to impose.

In our first main theorem, we will characterize additions on $\mathcal{K}_0^n$ by their induced homothety operation. In order to explain this point of view, notice that for any addition $\oplus$ on $\mathcal{K}_0^n$, we may define a corresponding homothety operation: for every natural number m we define

$$m \odot A = \underbrace{A \oplus A \oplus \cdots \oplus A}_{mtimes}.$$

Let us denote the homothety corresponding to $+_p$ by $\cdot_p$. Remember that we also defined

$$\lambda A = \{\lambda x : \ x \in A\}$$

for any $\lambda > 0$ (which may or may not be a natural number). It is straightforward to check that for every p we have

$$m \cdot_p A = m^{\frac{1}{p}} A. \tag{2.1}$$

The first theorem of this note proves that for every p equation 2.1 characterizes the p-addition uniquely. In fact, we will prove something a bit stronger:

Theorem 2.2. *Let $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ be an addition operation. Assume there exists a function $f : \mathbb{N} \to \mathbb{R}_+$ such that $m \odot A = f(m)A$ for every $A \in \mathcal{K}_0^n$ and every integer m. Then:*

(1) If f is not the constant function 1, then there exists $p \neq 0$ such that $A \oplus B = A +_p B$ for every $A, B \in \mathcal{K}_0^n$. If $n \ge 2$ then $1 \le |p| < \infty$.
(2) If $f \equiv 1$ and the identity element of $\oplus$ is $\{0\}$, then $A \oplus B = A +_\infty B$ for every $A, B \in \mathcal{K}_0^n$. Similarly, if the identity element is $\mathbb{R}^n$ then $A \oplus B = A +_{(-\infty)} B$.

(3) In dimension $n = 1$ there exists an addition $\oplus$ on $\mathcal{K}_s^1$ for which $f \equiv 1$, but the identity element of $\oplus$ is neither $\{0\}$ nor $\mathbb{R}$.

We do not know if an example similar to (3) exists in dimension $n \geq 2$.

It is interesting to notice that even though we never assumed $\oplus$ is commutative, we obtain this fact as a corollary of the theorem. A similar phenomenon appeared in the results of Gardner, Hug and Weil such as Theorem 2.1 that we mentioned before.

In the next section we will prove the theorem under the assumption that $f(2) > 1$. The remaining cases will be deduced in the short section 4. Let us mention that a straightforward modification of our proof will give the same result for additions on origin symmetric (not necessarily compact) convex bodies.

Our next goal is to characterize the Minkowski addition using Minkowski's polynomiality theorem. In fact, we will assume a little less:

Definition 2.6. We say that $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ is polynomial if for every $A, B \in \mathcal{K}_0^n$ we can write

$$|(s \odot A) \oplus (t \odot B)| = \sum_{i,j=0}^{d} c_{ij} s^i t^j$$

for all integers $s, t \in \mathbb{N}$. Here $d = d(A, B) \in \mathbb{N}$ is some number which may depend on A and B, and $c_{ij} = c_{ij}(A, B) \in (-\infty, \infty]$ are some coefficients that also depend on A and B.

Note that we must allow the coefficients to attain the value $+\infty$. We also allow them to be negative, but we do not allow the value $-\infty$, as this may lead to ambiguities such as $\infty - \infty$. Definition 2.6 has a similar counterpart in [4]. However, the definitions are not equivalent, since we are using the induced homothety $\odot$ in our definition.

We want to prove that the Minkowski sum $+$ (or, to be exact, the 1-sum $+_1$), is the only polynomial addition on $\mathcal{K}_0^n$. We can do this under a few weak additional assumptions:

Theorem 2.3. *Fix $n \geq 2$. Assume $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ is a polynomial addition operation. Assume further that:*

(1) $\{0\}$ is the identity element of $\oplus$: $A \oplus \{0\} = \{0\} \oplus A$ for every $A \in \mathcal{K}_0^n$.
(2) $\oplus$ is divisible: For every $A \in \mathcal{K}_0^n$ and every integer m there exists $B \in \mathcal{K}_0^n$ such that $m \odot B = A$.
(3) If $m \odot A \subseteq m \odot B$ for some integer m, then $A \subseteq B$.

Then either $A \oplus B = A +_\infty B$ (and the polynomial is just constant) or $A \oplus B = A +_1 B$.

Theorem 2.3 will be proven in section 5. We will also prove there another characterization theorem for p-sums, which seems to be interesting by its own right.

Let us conclude this introduction with three remarks concerning Definition 2.1 and our notion of an "addition operation":

Remark 2.1. Almost all interesting additions on convex bodies are indeed "addition operations" in the formal sense of Definition 2.1. However, there is at least one important exception, which is the Blaschke addition. We will not define this addition here, and instead refer the interested reader to section 8.2.2 of [12]. Since the Blaschke sum is not monotone, it is formally not an "addition operation", so our results will not cover this case.

A characterization of the Blaschke sum was recently obtain by Gardner, Parapatits and Schuster (see [5]).

Remark 2.2. In the definition of an addition operation we asked for $\oplus$ to have an identity element. This is necessary in order to exclude some pathologies, such as the addition

$$A \oplus B = \begin{cases} \{0\} & A = \{0\} \text{ or } B = \{0\} \\ A +_1 B & \text{otherwise.} \end{cases}$$

Note, however, that for different additions we may have different identity elements. For example, if $p \geq 1$ then the identity element of $+_p$ is $\{0\}$, while if $p \leq -1$ the identity element of $+_p$ is $\mathbb{R}^n$.

Remark 2.3. In Theorem 2.3 we assume that $\{0\}$ is the identity element with respect to $\oplus$. It may be interesting to notice that this algebraic condition is equivalent to a certain monotonicity condition: $\{0\}$ is the identity element of $\oplus$ if and only if $A \subseteq A \oplus B$ for every $A, B \in \mathcal{K}_0^n$.

Of course, if we know that our addition satisfies this stronger assumption, then we may also improve Theorem 2.2:

Corollary 2.1. *Let $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ be an addition operation such that $A \subseteq A \oplus B$ for all $A, B \in \mathcal{K}_0^n$. Assume there exists a function $f : \mathbb{N} \to \mathbb{R}_+$ such that $n \odot A = f(n)A$ for every $A \in \mathcal{K}_0^n$ and every integer n. Then there exists $0 < p \leq \infty$ such that $A \oplus B = A +_p B$ for every $A, B \in \mathcal{K}_0^n$. If $n \geq 2$ then $1 \leq p < \infty$.*

A similar corollary can be stated for negative values of p, using the "dual" condition $A \supseteq A \oplus B$.

3. Theorem 2.2 in the case $f(2) > 1$

In this section we will prove the following case of Theorem 2.2:

Proposition 3.1. *Let $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ be an addition operation. Assume there exists a function $f : \mathbb{N} \to \mathbb{R}_+$ such that $m \odot A = f(m)A$ for every $A \in \mathcal{K}_0^n$ and every integer m. If $f(2) > 1$, then there exists $p > 0$ such $A \oplus B = A +_p B$ for every $A, B \in \mathcal{K}_0^n$. If $n \geq 2$ then $p \geq 1$.*

We will now prove the theorem, by a sequence of claims:

Claim 3.1. $\{0\}$ *is the identity element with respect to* $\oplus$.

Proof. Denote the identity element by K. If $K \neq \{0\}$ there exists $0 \neq a \in K$, and then by convexity $[0,a] \subseteq K$. But then we get from monotonicity that

$$[0,a] = [0,a] \oplus K \supseteq [0,a] \oplus [0,a] = f(2)\,[0,a] = [0, f(2) \cdot a]\,.$$

Since $f(2) > 1$, this is obviously a contradiction. □

Claim 3.2. *There exists* $q > 0$ *such that* $f(m) = m^q$.

Proof. First, we prove that f is monotone increasing. Fix some compact set $\{0\} \neq K \in \mathcal{K}_0^n$, and notice that for any n we have

$$f(m+1)K = (m+1) \odot K = (m \odot K) \oplus K \supseteq (m \odot K) \oplus \{0\} = m \odot K = f(m)K.$$

It follows that indeed $f(m+1) \geq f(m)$.

Next, we prove that f is multiplicative: For all integers m and k we have

$$f(mk)K = (mk) \odot K = m \odot (k \odot K) = m \odot (f(k)K) = f(m)f(k)K,$$

so $f(mk) = f(m)f(k)$.

However, it is known that every increasing and multiplicative function must be of the form $f(m) = m^q$, so we are done. This follows for example from a more general theorem of Erdős ([1]), and a short and accessible proof of the exact result we use appears for example in [6]. □

From now on we will write $p = \frac{1}{q}$. Our goal is to prove that $p \geq 1$, and that $A \oplus B = A +_p B$ for every $A, B \in \mathcal{K}_0^n$.

Let us write

$$M_p(a,b) = (a^p + b^p)^{\frac{1}{p}}$$

for every $0 < p < \infty$ and $0 \leq a, b \leq \infty$. For every $\theta \in S^{n-1}$ and $c \in \mathbb{R}$ let us also define

$$H_{\theta,c} = \{x \in \mathbb{R}^n : \ \langle x, \theta \rangle \leq c\}\,.$$

We will simply write H_θ for $H_{\theta,0}$. Of course, $H_{\theta,\infty}$ is $\mathbb{R}^n$ itself.

Claim 3.3. *For every* $\theta \in S^{n-1}$ *and every* $0 \leq c, d \leq \infty$ *we have* $H_{\theta,c} \oplus H_{\theta,d} = H_{\theta, M_p(c,d)}$

Proof. First assume that $c^p = \frac{m}{k}$ and $d^p = \frac{s}{t}$ are positive rationals, then

$$\begin{aligned} H_{\theta,c} \oplus H_{\theta,d} &= \left(\frac{m^q}{k^q} H_{\theta,1}\right) \oplus \left(\frac{s^q}{t^q} H_{\theta,1}\right) \\ &= \left[(mt) \odot \left(\frac{1}{k^q t^q} H_{\theta,1}\right)\right] \oplus \left[(sk) \odot \left(\frac{1}{k^q t^q} H_{\theta,1}\right)\right] \\ &= (mt + sk) \odot \left(\frac{1}{k^q t^q} H_{\theta,1}\right) = \left(\frac{mt+sk}{kt}\right)^q H_{\theta,1} \\ &= \left(\frac{m}{k} + \frac{s}{t}\right)^q H_{\theta,1} = \left(c^p + d^p\right)^{\frac{1}{p}} H_{\theta,1} = H_{\theta, M_p(c,d)}. \end{aligned}$$

Since the rationals are dense in $[0, \infty]$, all the remaining cases can be proven by approximation, using the monotonicity of $\oplus$. □

Claim 3.4. *For every $A \in \mathcal{K}_0^n$ we have $A \oplus H_\theta = H_\theta \oplus A = H_{\theta, h_A(\theta)}$.*

Proof. We will only prove that $A \oplus H_\theta = H_{\theta, h_A(\theta)}$. The proof that $H_\theta \oplus A = H_{\theta, h_A(\theta)}$ is completely analogous.

For one inclusion, notice that $A \oplus H_\theta \supseteq A \oplus \{0\} = A$, and similarly $A \oplus H_\theta \supseteq H_\theta$. Since $A \oplus H_\theta$ is convex and closed we must have

$$A \oplus H_\theta \supseteq \overline{\operatorname{conv}\{A, H_\theta\}} = H_{\theta, h_A(\theta)}.$$

Next we prove the opposite inclusion. We know that $A \subseteq H_{\theta, h_A(\theta)}$, so by monotonicity and claim 3.3 we see that

$$A \oplus H_\theta \subseteq H_{\theta, h_A(\theta)} \oplus H_{\theta,0} = H_{\theta, M_p(h_A(\theta), 0)} = H_{\theta, h_A(\theta)}$$ □

Claim 3.5. *For every $A, B \in \mathcal{K}_0^n$ we have $A \oplus B = A +_p B$.*

Proof. Fix $\theta \in S^{n-1}$. Our goal is to prove that $h_{A \oplus B}(\theta) = M_p(h_A(\theta), h_B(\theta))$.

On the one hand, using the previous claim, we know that

$$(A \oplus B) \oplus H_\theta = H_{\theta, h_{A \oplus B}(\theta)}.$$

On the other hand, since $H_\theta \oplus H_\theta = H_\theta$, we may write

$$\begin{aligned} (A \oplus B) \oplus H_\theta &= (A \oplus B) \oplus (H_\theta \oplus H_\theta) = A \oplus (B \oplus H_\theta) \oplus H_\theta \\ &= A \oplus (H_\theta \oplus B) \oplus H_\theta = (A \oplus H_\theta) \oplus (B \oplus H_\theta) = H_{\theta, h_A(\theta)} \oplus H_{\theta, h_B(\theta)} \\ &= H_{\theta, M_p(h_A(\theta), h_B(\theta))} \end{aligned}$$

This implies that indeed $h_{A \oplus B}(\theta) = M_p(h_A(\theta), h_B(\theta))$, and the proof is complete. □

The only thing that remains to be proven is that indeed $p \geq 1$ when the dimension n is at least 2. This is obvious, as it is well known that the p-sum does not preserve convexity for $0 < p < 1$. For completeness, let us give a short proof for this simple fact:

Claim 3.6. *If $n \geq 2$ then we must have $p \geq 1$.*

Proof. Choose $\{e_1, e_2\} \subseteq \mathbb{R}^n$ to be an orthonormal pair. Choose $A = [0, e_1]$, $B = [0, e_2]$, and $C = A \oplus B = A +_p B$.

A simple calculation gives:

$$\begin{aligned} h_C(e_1) &= (h_A(e_1)^p + h_B(e_1)^p)^{\frac{1}{p}} = 1 \\ h_C(e_2) &= (h_A(e_2)^p + h_B(e_2)^p)^{\frac{1}{p}} = 1 \\ h_C\left(\frac{e_1+e_2}{2}\right) &= \left(h_A\left(\frac{e_1+e_2}{2}\right)^p + h_B\left(\frac{e_1+e_2}{2}\right)^p\right)^{\frac{1}{p}} = 2^{\frac{1-p}{p}}. \end{aligned}$$

Since h_C is a convex function we see that

$$2^{\frac{1-p}{p}} = h_C\left(\frac{e_1+e_2}{2}\right) \leq \frac{h_C(e_1) + h_C(e_2)}{2} = 1,$$

which implies that $p \geq 1$. □

4. The remaining cases of Theorem 2.2

Using Proposition 3.1 and duality, we can also understand the situation $f(2) < 1$.

Proposition 4.1. *Let $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ be an addition. Assume there exists a function $f : \mathbb{N} \to \mathbb{R}_+$ such that $m \odot A = f(m)A$ for every $A \in \mathcal{K}_0^n$ and every integer m. If $f(2) < 1$, then there exists $p > 0$ such $A \oplus B = A +_{-p} B$ for every $A, B \in \mathcal{K}_0^n$. If $n \geq 2$ then $p \geq 1$.*

Proof. Define a new addition $\boxplus$ on $\mathcal{K}_0^n$ by

$$A \boxplus B = (A^\circ \oplus B^\circ)^\circ.$$

It is easy to check that if $\oplus$ is an addition operation with identity K, then $\boxplus$ is also an addition with identity K°.

Notice that for every $A \in \mathcal{K}_0^n$ and integer m we have

$$m \boxdot A = (m \odot A^\circ)^\circ = (f(m)A^\circ)^\circ = \frac{1}{f(m)} A^{\circ\circ} = \frac{1}{f(m)} A.$$

Since $\frac{1}{f(2)} > 1$, we may apply Proposition 3.1 and deduce that we have $p > 0$ (or $p \geq 1$ when $n \geq 2$) such that $A \boxplus B = A +_p B$ for all $A, B \in \mathcal{K}_0^n$. But then

$$A \oplus B = (A^\circ \boxplus B^\circ)^\circ = (A^\circ +_p B^\circ)^\circ = A +_{-p} B,$$

which completes the proof. □

Finally we must deal with the case $f(2) = 1$:

Proposition 4.2. *Let $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ be an addition. Assume that $A \oplus A = A$ for every $A \in \mathcal{K}_0^n$.*

(1) If the identity element of $\oplus$ is $\{0\}$, then $A \oplus B = A +_\infty B$ for every $A, B \in \mathcal{K}_0^n$.
(2) If the identity element is $\mathbb{R}^n$ then $A \oplus B = A +_{(-\infty)} B$.

Proof. Assume first that $\{0\}$ is the identity element with respect to $\oplus$.

Then for every $A, B \in \mathcal{K}_0^n$ we know that

$$A \oplus B \supseteq A \oplus \{0\} = A$$
$$A \oplus B \supseteq \{0\} \oplus B = B,$$

and since $A \oplus B$ is closed and convex we must have $A \oplus B \supseteq A +_\infty B$. On the other hand

$$A \oplus B \subseteq (A +_\infty B) \oplus (A +_\infty B) = A +_\infty B,$$

so the proof is complete.

The case that $\mathbb{R}^n$ is the identity element is handled in the same way. $\square$

Finally, let us give the promised example for symmetric bodies in dimension 1:

Example 4.1. Define $\oplus : \mathcal{K}_s^1 \times \mathcal{K}_s^1 \to \mathcal{K}_s^1$ by

$$[-a, a] \oplus [-b, b] = \begin{cases} [-\min\{a, b\}, \min\{a, b\}] & a \leq 1 \text{ and } b \leq 1 \\ [-\max\{a, b\}, \max\{a, b\}] & \text{otherwise.} \end{cases}$$

It is obvious that $\oplus$ is monotone. It is also straightforward, though a bit tedious, to check that $\oplus$ is associative.

The identity element of $\oplus$ is of course $[-1, 1]$, which is neither $\{0\}$ nor $\mathbb{R}$.

This completes the proof of Theorem 2.2.

5. Characterizing the addition by Minkowski's theorem

Our main goal in this section is to prove Theorem 2.3. We start with a slightly different characterization theorem, that seems to be interesting by its own right:

Theorem 5.1. *Fix $n \geq 2$. Assume $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ is an addition operation. Assume further that:*

(1) $\{0\}$ is the identity element of $\oplus$: $A \oplus \{0\} = \{0\} \oplus A$ for every $A \in \mathcal{K}_0^n$.
(2) $\oplus$ is divisible: For every $A \in \mathcal{K}_0^n$ and every integer m there exists $B \in \mathcal{K}_0^n$ such that $m \odot B = A$.
(3) If $m \odot A \subseteq m \odot B$ for some integer m, then $A \subseteq B$.
(4) For every subspace V of $\mathbb{R}^n$ we have $V \oplus V = V$.

Then there exists $1 \leq p \leq \infty$ such that $A \oplus B = A +_p B$ for all $A, B \in \mathcal{K}_0^n$.

In the proof of Theorem 5.1 we will reuse some of the claims we had in Section 3. However, we will also need two new claims:

Claim 5.1. *For every $A, B \in \mathcal{K}_0^n$ and every integer m we have $m \odot (A \cap B) = (m \odot A) \cap (m \odot B)$.*

Proof. One inclusion is immediate from monotonicity: $A \cap B \subseteq A$ implies $m \odot (A \cap B) \subseteq m \odot A$. Similarly $m \odot (A \cap B) \subseteq m \odot B$, so we see that indeed

$$m \odot (A \cap B) \subseteq (m \odot A) \cap (m \odot B) .$$

For the second inclusion, by condition 2 there exists $C \in \mathcal{K}_0^n$ such that

$$m \odot C = (m \odot A) \cap (m \odot B) .$$

Since $m \odot C \subseteq m \odot A$, condition 3 implies that $C \subseteq A$. Similarly $C \subseteq B$, and then $C \subseteq A \cap B$ so

$$(m \odot A) \cap (m \odot B) = m \odot C \subseteq m \odot (A \cap B).$$

This completes the proof. □

Claim 5.2. *For every $m \in \mathbb{N}$ there exists a number $f(m) \geq 1$ such that*

$$m \odot H_{\theta,c} = f(m) H_{\theta,c} = H_{\theta, f(m)c}$$

for all $\theta \in S^{n-1}$ and $c > 0$.

Proof. Note that by monotonicity and condition 1 we have

$$m \odot H_{\theta,c} \supseteq [(m-1) \odot \{0\}] \oplus H_{\theta,c} = H_{\theta,c},$$

So $m \odot H_{\theta,c} = H_{\theta,\lambda c}$ for some $\lambda \geq 1$. Our goal is to prove that λ is independent of θ and c.

So, assume that $m \odot H_{\theta,c} = H_{\theta,\lambda c}$ and $m \odot H_{\eta,d} = H_{\eta,\mu d}$. Our goal is to prove that $\lambda = \mu$, and we may assume that $\theta \neq \eta$. This means that we can find a point $x_0 \in \mathbb{R}^n$ such that $\langle x_0, \theta \rangle = c$ and $\langle x_0, \eta \rangle = d$. If we define $A = (-\infty, x_0]$ to be the ray emanating from x_0 and passing through the origin, then

$$A = H_{\theta,c} \cap \mathbb{R}x_0 = H_{\eta,d} \cap \mathbb{R}x_0.$$

Now we apply the previous claim to $H_{\theta,c}$ and $\mathbb{R}x_0$ and see that

$$m \odot A = m \odot (H_{\theta,c} \cap \mathbb{R}x_0) = (m \odot H_{\theta,c}) \cap (m \odot \mathbb{R}x_0) = H_{\theta,\lambda c} \cap \mathbb{R}x_0 = (-\infty, \lambda x_0].$$

Of course, we used condition 4 to deduce that $m \odot \mathbb{R}x_0 = \mathbb{R}x_0$.

But exactly the same reasoning shows us that

$$m \odot A = m \odot (H_{\eta,d} \cap \mathbb{R}x_0) = (m \odot H_{\eta,d}) \cap (m \odot \mathbb{R}x_0) = H_{\eta,\mu d} \cap \mathbb{R}x_0 = (-\infty, \mu x_0]$$

This shows that $\lambda = \mu$ as we wanted. □

Once we have the above claims, the proof of Theorem 5.1 is almost immediate:

Proof. We know that there exists $f : \mathbb{N} \to [1, \infty)$ such that $m \odot H_{\theta,c} = f(m) H_{\theta,c}$ for all half spaces $H_{\theta,c}$. If $f(2) > 1$, then the proof of Claim 3.2 shows that $f(m) =$

$m^{\frac{1}{p}}$ for some $0 < p < \infty$. The rest of the proof proceeds exactly like the proof of Proposition 3.1.

If $f(2) = 1$, then $H \oplus H = H$ for every half space H. Let us show that the same is true for every $A \in \mathcal{K}_0^n$: On the one hand

$$A \oplus A \supseteq A \oplus \{0\} = A.$$

On the other hand for every H which supports A we have

$$A \oplus A \subseteq H \oplus H = H.$$

Since every closed convex set is uniquely defined by its supporting hyperplanes, we see that $A \oplus A \subseteq A$.

Since $A \oplus A = A$ for all $A \in \mathcal{K}_0^n$, we may apply Proposition 4.2 and conclude that $\oplus = +_\infty$. ☐

Now we are ready to prove Theorem 2.3. We want to use Theorem 5.1 in the proof. However, in the statement of Theorem 2.3 we did not assume condition 4 of Theorem 5.1, so we have to prove it:

Lemma 5.1. *Under the assumptions of theorem 2.3, we must have $V \oplus V = V$ for every subspace V of $\mathbb{R}^n$.*

Proof. Let V be any fixed subspace of $\mathbb{R}^n$. We always have $V \oplus V \supseteq V \oplus \{0\} = V$.

For the other inclusion assume by contradiction that $a \in V \oplus V$ but $a \notin V$, we first assume that $\dim V = n - 1$. Notice that in this case

$$V \oplus V \supseteq \overline{\operatorname{conv}\{V, a\}},$$

which is a set of infinite volume. However, by polynomiality

$$|s \odot V| = |(s \odot V) \oplus (1 \odot \{0\})| = \sum_{i,j=0}^{d} c_{ij} s^i 1^j = \sum_{i=0}^{d} \widetilde{c_i} s^i.$$

Plugging in $s = 2$ we see that

$$\sum_{i=0}^{d} \left(\widetilde{c_i} \cdot 2^i\right) = |V \oplus V| = \infty,$$

so $\widetilde{c_i} = \infty$ for some i. But then we will also have

$$|V| = |1 \odot V| = \sum_{i=0}^{d} \widetilde{c_i} = \infty.$$

This is a contradiction, since a proper subspace always satisfy $|V| = 0$.

Now we assume that $\dim V = m < n - 1$. We may choose our coordinate system in such a way that

$$V = \{(x_1, x_2, \ldots, x_n) : \; x_{m+1} = x_{m+2} = \cdots = x_n = 0\}.$$

and $a = e_{m+1} = (0,0,\dots,0,1,0,\dots,0)$. Since $a \in V \oplus V$ we must have

$$V \oplus V \supseteq \overline{\operatorname{conv}\{V,a\}} = \left\{ (x_1,x_2,\dots,x_n) : \begin{array}{l} 0 \le x_{m+1} \le 1 \\ x_{m+2} = x_{m+3} = \cdots = x_n = 0 \end{array} \right\}.$$

Now let us define

$$W = \{(x_1,x_2,\dots,x_n) :\ x_1 = x_2 = \cdots = x_{m+1} = 0\}.$$

Notice that

$$V \oplus W \subseteq 2 \odot \{(x_1,x_2,\dots,x_n) :\ x_{m+1} = 0\} = \{(x_1,x_2,\dots,x_n) :\ x_{m+1} = 0\},$$

so $|V \oplus W| = 0$. However,

$$V \oplus V \oplus W \supseteq \overline{\operatorname{conv}\{V,a,W\}} = \{(x_1,x_2,\dots,x_n) :\ 0 \le x_{m+1} \le 1\},$$

so $|2 \odot V \oplus W| = \infty$. This will give contradiction to the polynomiality of $|(s \odot V) \oplus (t \odot W)|$, just like the previous case. □

Now we can prove Theorem 2.3:

Proof. By the lemma, $\oplus$ satisfies all conditions of Theorem 5.1, so $\oplus = +_p$ for some $1 \le p \le \infty$. We only have to check that for $1 < p < \infty$ there will be no polynomiality. This is just a computation, which is similar (though not identical) to the one in [4]. Indeed, define $q \in (1,\infty)$ by $\frac{1}{q} + \frac{1}{p} = 1$ and define

$$K = \left\{ x \in \mathbb{R}^n :\ \|x\|_q \le 1 \right\}.$$

Remember that $h_K(\theta) = \|\theta\|_p$.

Let us choose as our convex bodies $A = T(K)$ and $B = S(K)$, where $T, S \in GL(n)$ are positive diagonal matrices: $T = \operatorname{diag}(t_1,t_2,\dots,t_n)$ and $S = \operatorname{diag}(s_1,s_2,\dots,s_n)$ for $s_i, t_i > 0$.

We want to understand what is the body $C = (m \odot A) \oplus (k \odot B)$. For every $\theta = (\theta_1,\dots,\theta_n)$ we have

$$\begin{aligned} h_C(\theta) &= \left(m \cdot h_A(\theta)^p + k \cdot h_B(\theta)^p\right)^{\frac{1}{p}} = \left(m \cdot h_K(T^*\theta)^p + k \cdot h_K(S^*\theta)^p\right)^{\frac{1}{p}} \\ &= \left(m \|T\theta\|_p^p + k \|S\theta\|_p^p\right)^{\frac{1}{p}} = \left\| (mT^p + kS^p)^{\frac{1}{p}} \theta \right\|_p, \end{aligned}$$

so

$$(m \odot A) \oplus (k \odot B) = (mT^p + kS^p)^{\frac{1}{p}} K.$$

It follows that

$$|(m \odot A) \oplus (k \odot B)| = |K| \cdot \det (mT^p + kS^p)^{\frac{1}{p}},$$

so $\det (mT^p + kS^p)^{\frac{1}{p}}$ must be a polynomial in m and k for every choice of diagonal matrices T and S.

From here it is simple to obtain a contradiction: by taking $T = S = Id$ we get that

$$\det (mI + kI)^{\frac{1}{p}} = (m+k)^{\frac{n}{p}}$$

is a polynomial in m and k. By taking $T = \mathrm{diag}(2^{\frac{1}{p}}, 1, 1, \dots, 1)$ and $S = Id$ we get that

$$\det(mT^p + kI)^{\frac{1}{p}} = \det\left(\mathrm{diag}(2m+k, m+k, \dots, m+k)^{\frac{1}{p}}\right) = (2m+k)^{\frac{1}{p}} (m+k)^{\frac{n-1}{p}}$$

is also a polynomial. Hence their quotient

$$\frac{(m+k)^{\frac{n}{p}}}{(2m+k)^{\frac{1}{p}} (m+k)^{\frac{n-1}{p}}} = \left(\frac{m+k}{2m+k}\right)^{\frac{1}{p}}$$

is a rational function. This is impossible, since $0 < \frac{1}{p} < 1$. □

Let us conclude this section by mentioning a possible alternative formulations for Theorems 2.3 and 5.1:

(1) As already mentioned in Remark 2.3, condition 1 is equivalent to the statement that $\oplus$ has some identity element K, and that $A \subseteq A \oplus B$ for all $A, B \in \mathcal{K}_0^n$.
(2) Notice that in the proof of Theorem 5.1 we only used conditions 2 and 3 in the proof of Claim 5.1. Hence we may replace these assumptions by the single assumption that

$$m \odot (A \cap B) = (m \odot A) \cap (m \odot B)$$

for every $A, B \in \mathcal{K}_0^n$ and every integer m.
(3) It is possible to replace condition 2 by the requirement $\oplus$ is continuous in the Hausdorff sense. To do this one disposes of Claim 5.1, and proves Claim 5.2 directly using a slightly different argument. We will not give all the details here.

6. Dealing with compact sets

So far we have stated and proved all our theorems for the class $\mathcal{K}_0^n$, which also contains non-compact sets. These non-compact sets, such as half spaces, were heavily used in many of the proofs. However, it is sometimes more natural to consider instead the class $\mathcal{K}_{0,c}^n$, which is the class of all compact bodies inside $\mathcal{K}_0^n$. In this final section we will explain how our results can be proved for this class as well.

The main idea is that under a weak regularity condition, additions on $\mathcal{K}_{0,c}^n$ may be extended to additions on $\mathcal{K}_0^n$ with the same properties. We say that $\oplus$ is an addition on $\mathcal{K}_{0,c}^n$ if it satisfies the obvious analog of Definition 2.1. The exact condition we will impose on $\oplus$ is the following:

Definition 6.1. Fix a family of sets $\{A_m\}_{m=1}^{\infty} \subseteq \mathcal{K}_{0,c}^n$ and set $A \in \mathcal{K}_{0,c}^n$. We write $A_m \nearrow A$ if

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

and

$$A = \overline{\bigcup_{m=1}^{\infty} A_m}.$$

Definition 6.2. Let $\oplus : \mathcal{K}_{0,c}^n \times \mathcal{K}_{0,c}^n \to \mathcal{K}_{0,c}^n$ be any map. We say that $\oplus$ is continuous from below if whenever we have $A_m \nearrow A$ and $B_m \nearrow B$ we also have $(A_m \oplus B_m) \nearrow (A \oplus B)$.

We want to show that every addition operation $\oplus$ on $\mathcal{K}_{0,c}^n$ which is continuous from below may be extended to an addition on $\mathcal{K}_0^n$. For the construction we will need the following piece of notation: given $A \in \mathcal{K}_0^n$, we say that $P \preceq A$ if:

(1) P is a polytope.
(2) P is contained in the relative interior of A (i.e. the interior of A with respect to its affine hull).
(3) $0 \in P$.

Given $\oplus : \mathcal{K}_{0,c}^n \times \mathcal{K}_{0,c}^n \to \mathcal{K}_{0,c}^n$, we define an operation $\boxplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ by

$$A \boxplus B = \overline{\bigcup \{P \oplus Q : \ P \preceq A, \ Q \preceq B\}}.$$

The first result about $\boxplus$ is the following:

Proposition 6.1. *Assume $\oplus$ is an addition operation on $\mathcal{K}_{0,c}^n$ which is continuous. Then $\boxplus$ is an addition operation on $\mathcal{K}_0^n$ which extends $\oplus$.*

Proof. First we show that $A \boxplus B$ is indeed a convex set, so $\boxplus$ is well defined. Of course, it is enough to prove that

$$C = \bigcup \{P \oplus Q : \ P \preceq A, \ Q \preceq B\}$$

is convex (without the closure). Fix $x, y \in C$. Then $x \in P_1 \oplus Q_1$ and $y \in P_2 \oplus Q_2$ for some $P_i \preceq A$ and $Q_i \preceq B$. If we now define

$$P = P_1 +_\infty P_2, \qquad Q = Q_1 +_\infty Q_2,$$

then $P \preceq A$, $Q \preceq B$ and by monotonicity $x, y \in P \oplus Q$. Since $P \oplus Q$ is convex we see that $[x, y] \subseteq P \oplus Q \subseteq C$, so C is indeed convex.

Next we show that $\boxplus$ is an extension of $\oplus$, i.e. $A \boxplus B = A \oplus B$ for every $A, B \in \mathcal{K}_{0,c}^n$. The inclusion ($\subseteq$) follows from the monotonicity of $\oplus$. For the opposite inclusion, just choose sequences $\{P_m\}, \{Q_m\}$ such that $P_m \preceq A$, $Q_m \preceq B$, and $P_m \nearrow A$, $Q_m \nearrow B$. From the definition we will have $A \boxplus B \supseteq P_m \oplus Q_m$ for all n, and since $(P_m \oplus Q_m) \nearrow (A \oplus B)$, the claim follows.

Next, we need to prove that $\boxplus$ is an addition operation. Monotonicity is clear from the definition. Assume $K \in \mathcal{K}_{0,c}^n$ is the identity element of $\oplus$. Then for every $P \preceq A$ and $Q \preceq K$ we have

$$P \oplus Q \subseteq P \oplus K = P \subseteq A,$$

so $A \boxplus K \subseteq A$. For the opposite direction we again take sequences $\{P_m\}, \{Q_m\}$ such that $P_m \preceq A$, $Q_m \preceq K$, and $P_m \nearrow A$, $Q_m \nearrow K$. For every integers m and k we have $A \boxplus K \supseteq P_m \oplus Q_k$. Fixing m and sending $k \to \infty$, we know that $(P_m \oplus Q_k) \nearrow (P_m \oplus K) = P_m$, so $A \boxplus K \supseteq P_m$. Since this was true for all m we have $A \boxplus K \supseteq A$.

Finally, we need to prove that $\boxplus$ is associative. In fact, we will prove that

$$(A \boxplus B) \boxplus C = \overline{\bigcup \{(P \oplus Q) \oplus R : \ P \preceq A, \ Q \preceq B, \ R \preceq C\}}. \tag{6.1}$$

This, and the analog equality for $A \boxplus (B \boxplus C)$, will prove the result we want.

The inclusion $(\supseteq)$ is easy, since we have already seen that $\boxplus$ is monotone and extends $\oplus$. Hence for every $P \preceq A$, $Q \preceq B$ and $R \preceq C$ we have

$$(P \oplus Q) \oplus R = (P \boxplus Q) \boxplus R \subseteq (A \boxplus B) \boxplus C,$$

and the inclusion follows.

For the opposite inclusion, fix $S \preceq A \boxplus B$ and $R \preceq C$. Since S is a polytope, we may write $S = \text{conv}\{x_1, x_2, \dots, x_m\}$, and every x_i is in the interior of $A \boxplus B$. By definition, this means that $x_i \in P_i \oplus Q_i$ for some $P_i \preceq A$ and $Q_i \preceq B$. Taking

$$P = P_1 +_\infty P_2 +_\infty \cdots +_\infty P_m$$
$$Q = Q_1 +_\infty Q_2 +_\infty \cdots +_\infty Q_m,$$

we see that $P \preceq A$, $Q \preceq B$, and $S \subseteq P \oplus Q$. Hence we have

$$S \oplus R \subseteq (P \oplus Q) \oplus R,$$

so the other inclusion follows and our proof is complete. □

The use of polytopes in the definition of $\boxplus$ made the proof of associativity simpler. It is easy to check that in fact we also have

$$A \boxplus B = \overline{\bigcup \left\{K \oplus T : \ K, T \in \mathcal{K}^n_{0,c}, \ K \subseteq A, \ T \subseteq B\right\}},$$

but we will not need this fact.

In order to have an analog of Theorem 2.2, we need to relate the homotheties of $\boxplus$ to the homotheties of $\oplus$:

Proposition 6.2. *For every $A \in \mathcal{K}^n_0$ and $m \in \mathbb{N}$ we have*

$$m \boxdot A = \overline{\bigcup \{m \odot P : \ P \preceq A\}}.$$

Proof. Extending equation (6.1) in the obvious way to sums of m sets we give

$$m \boxdot A = \overline{\bigcup \{P_1 \oplus P_2 \oplus \cdots \oplus P_m : \ P_1, P_2, \dots, P_m \preceq A\}}.$$

From here the inclusion $(\supseteq)$ is obvious. For the opposite inclusion, just define

$$P = P_1 +_\infty P_2 +_\infty \cdots +_\infty P_m$$

and notice that by monotonicity $P_1 \oplus P_2 \oplus \cdots \oplus P_m \subseteq m \odot P$. □

We are now ready to prove the first main theorem of this section:

Theorem 6.1. *Let $\oplus : \mathcal{K}^n_{0,c} \times \mathcal{K}^n_{0,c} \to \mathcal{K}^n_{0,c}$ be an addition operation which is continuous from below. Assume there exists a function $f : \mathbb{N} \to \mathbb{R}_+$ such that $m \odot A = f(m)A$ for every $A \in \mathcal{K}^n_{0,c}$ and every integer m. Then:*

(1) If f is not the constant function 1, then there exists $p > 0$ such that $A \oplus B = A +_p B$ for every $A, B \in \mathcal{K}^n_{0,c}$. If $n \geq 2$ then $1 \leq p < \infty$.
(2) If $f \equiv 1$ and the identity element of $\oplus$ is $\{0\}$, then $A \oplus B = A +_\infty B$ for every $A, B \in \mathcal{K}^n_0$.

Notice that p-sums for $p < 0$ are automatically excluded, since their identity element, $\mathbb{R}^n$, is not a compact set.

Proof. The extension $\boxplus$ satisfies

$$\begin{aligned} m \boxdot A &= \overline{\bigcup \{m \odot P : \ P \preceq A\}} = \overline{\bigcup \{f(m)P : \ P \preceq A\}} \\ &= f(m)\overline{\bigcup \{P : \ P \preceq A\}} = f(m)A. \end{aligned}$$

To conclude, apply Theorem 2.2 to $\boxplus$. □

The extensions of Theorems 2.3 and 5.1 proceed in the same way. All we have to do is check that if $\oplus$ satisfies the conditions of these theorems, so does $\boxplus$. For example:

Proposition 6.3. *Assume $\oplus : \mathcal{K}^n_{0,c} \times \mathcal{K}^n_{0,c} \to \mathcal{K}^n_{0,c}$ is an addition operation which is continuous from below and satisfies:*

(1) $\oplus$ is divisible: For every $A \in \mathcal{K}^n_{0,c}$ and every integer m there exists $B \in \mathcal{K}^n_{0,c}$ such that $m \odot B = A$.
(2) If $m \odot A \subseteq m \odot B$ for some integer m, then $A \subseteq B$.

Then $\boxplus : \mathcal{K}^n_0 \times \mathcal{K}^n_0 \to \mathcal{K}^n_0$ satisfies the same two properties.

Proof. Notice that these two properties together imply for every $A \in \mathcal{K}^n_{0,c}$ there exists a *unique* $B \in \mathcal{K}^n_{0,c}$ such that $m \odot B = A$. Let us write $B = \frac{1}{m} \odot A$. We now define

$$\frac{1}{m} \boxdot A = \overline{\bigcup \left\{ \frac{1}{m} \odot P : \ P \preceq A \right\}} \tag{6.2}$$

and claim that $m \boxdot \left(\frac{1}{m} \boxdot A\right) = A$ for all $A \in \mathcal{K}^n_0$. By Proposition 6.2 we have

$$m \boxdot \left(\frac{1}{m} \boxdot A \right) = \overline{\bigcup \left\{ m \odot Q : \ Q \preceq \left(\frac{1}{m} \boxdot A \right) \right\}}.$$

For every $P \preceq A$ we have by definition $\frac{1}{m} \odot P \subseteq \frac{1}{m} \boxdot A$, so

$$P = m \odot \left(\frac{1}{m} \odot P\right) \subseteq m \boxdot \left(\frac{1}{m} \boxdot A\right).$$

This shows that indeed $A \subseteq m \boxdot \left(\frac{1}{m} \boxdot A\right)$.

For other inclusion, take $Q \preceq \frac{1}{m} \boxdot A$. Since Q is a polytope we can write $Q = \operatorname{conv}\{x_1, x_2, \ldots, x_k\}$, and for every $1 \le i \le k$ we have $x_i \in \frac{1}{m} \odot P_i$ for some $P_i \preceq A$. Taking

$$P = P_1 +_\infty P_2 +_\infty \cdots +_\infty P_k,$$

and using the monotonicity condition (2), we see that $Q \subseteq \frac{1}{m} \odot P$. Hence

$$m \odot Q \subseteq m \odot \left(\frac{1}{m} \odot P\right) = P \subseteq A,$$

and we see that indeed $m \boxdot \left(\frac{1}{m} \boxdot A\right) \subseteq A$. This proves that $\boxplus$ satisfies condition (1).

The fact that $\boxplus$ satisfies condition (2) is obvious from the explicit formula (6.2). □

This proposition, in turn, immediately implies:

Theorem 6.2. *Fix $n \ge 2$. Assume $\oplus : \mathcal{K}^n_{0,c} \times \mathcal{K}^n_{0,c} \to \mathcal{K}^n_{0,c}$ is an addition operation which is continuous from below. Assume further that:*

(1) $\{0\}$ is the identity element of $\oplus$: $A \oplus \{0\} = \{0\} \oplus A$ for every $A \in \mathcal{K}^n_{0,c}$.
(2) $\oplus$ is divisible: For every $A \in \mathcal{K}^n_{0,c}$ and every integer m there exists $B \in \mathcal{K}^n_{0,c}$ such that $m \odot B = A$.
(3) If $m \odot A \subseteq m \odot B$ for some integer m, then $A \subseteq B$.
(4) For every subspace V of $\mathbb{R}^n$ and every $A, B \in \mathcal{K}^n_{0,c}$ which satisfy $A, B \subseteq V$ we also have $A \oplus B \subseteq V$.

Then there exists $1 \le p \le \infty$ such that $A \oplus B = A +_p B$ for all $A, B \in \mathcal{K}^n_{0,c}$.

Proof. Using the previous proposition, we see that $\boxplus$ satisfies all conditions of Theorem 5.1. □

Finally, we want to have an variant of Theorem 2.3 about polynomiality of volume. We say that $\oplus : \mathcal{K}^n_{0,c} \times \mathcal{K}^n_{0,c} \to \mathcal{K}^n_{0,c}$ is polynomial if it satisfies the obvious analog of Definition 2.6. Since our sets are now compact they all have finite volume, so the coefficients c_{ij} of the polynomial are automatically finite as well. We can now state:

Theorem 6.3. *Fix $n \ge 2$. Assume $\oplus : \mathcal{K}^n_{0,c} \times \mathcal{K}^n_{0,c} \to \mathcal{K}^n_{0,c}$ is a polynomial addition operation which is continuous from below. Assume further that:*

(1) $\{0\}$ is the identity element of $\oplus$: $A \oplus \{0\} = \{0\} \oplus A$ for every $A \in \mathcal{K}^n_0$.

(2) $\oplus$ is divisible: For every $A \in \mathcal{K}_0^n$ and every integer m there exists $B \in \mathcal{K}_0^n$ such that $m \odot B = A$.
(3) If $m \odot A \subseteq m \odot B$ for some integer m, then $A \subseteq B$.

Then either $A \oplus B = A +_\infty B$ (and the polynomial is just constant) or $A \oplus B = A +_1 B$.

For the proof, note that there is no need to show that the extension $\boxplus$ is also polynomial, since the proof of Theorem 2.3 only used polynomiality for compact sets anyway. However, we do need to extend Lemma 5.1 to our case:

Lemma 6.1. *Assume $\oplus$ satisfies all the assumptions of Theorem 6.3. Then for every subspace V of $\mathbb{R}^n$ and every $A, B \in \mathcal{K}_{0,c}^n$ which satisfy $A, B \subseteq V$ we also have $A \oplus B \subseteq V$.*

Proof. The proof is similar to the proof of Lemma 5.1, so we will omit some of the details. By monotonicity, we may assume without loss of generality that

$$A = B = \{x \in V : \ |x| \leq R\}$$

for some $R > 0$. If, by contradiction, $A \oplus A \not\subseteq V$ then there exists $a \in (A \oplus A) \setminus V$.

First, assume that $\dim V = n - 1$. For every $s \in \mathbb{N}$ we may write

$$|s \odot A| = |(s \odot A) \oplus (1 \odot \{0\})| = \sum_{i,j=0}^{d} c_{ij} s^i 1^j = f(s),$$

for some polynomial f. Of course, since f is a polynomial, it is defined for every $x \in \mathbb{R}$, not only for natural numbers.

Fix $m \in \mathbb{N}$. By divisibility there exists $C \in \mathcal{K}_{0,c}^n$ such that $m \odot C = A$. By polynomiality of volume there exists a polynomial $g(x)$ such that $g(s) = |s \odot C|$ for every $s \in \mathbb{N}$. In particular we have for every $s \in \mathbb{N}$

$$f(s) = |s \odot A| = |s \odot (m \odot C)| = |(sm) \odot C| = g(ms).$$

We see that the polynomials $f(x)$ and $g(mx)$ coincide on $\mathbb{N}$, so we must have $f(x) = g(mx)$ for all $x \in \mathbb{R}$. Since $C \subseteq A \subseteq V$ we have $|C| = 0$, so

$$f\left(\frac{1}{m}\right) = g(1) = |C| = 0.$$

Since $f\left(\frac{1}{m}\right) = 0$ for every $m \in \mathbb{N}$, f must be the zero polynomial, and then $|s \odot A| = 0$ for all $s \in \mathbb{N}$.

However, $A \oplus A \supseteq \overline{\operatorname{conv}\{A, a\}}$, which is a set of positive volume. Hence $|2 \odot A| > 0$, we we arrived at a contradiction.

Now we assume $\dim V = m < n-1$. By choosing our coordinate system correctly, it is enough to consider the case

$$V = \{(x_1, x_2, \dots, x_n) : \ x_{m+1} = x_{m+2} = \cdots = x_n = 0\}$$

and $a = e_{m+1}$ (and A, like before, is $\{x \in V : |x| \le R\}$ for some R). If we now define

$$C = \left\{ x = (x_1, x_2, \ldots, x_n) : \begin{array}{l} x_1 = x_2 = \cdots = x_{m+1} = 0 \\ |x| \le 1 \end{array} \right\},$$

then $A \oplus C$ is of zero volume, while $A \oplus A \oplus C$ is of non-zero volume. This will give the same contradiction as before, for the polynomial $f(s) = |(s \odot A) \oplus C|$. $\square$

After we have the lemma, we can deduce Theorem 6.3 from Theorem 6.2, in exactly the same way that Theorem 2.3 was deduced from Theorem 5.1. No other changes in the proof are necessary.

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References

[1] Paul Erdös. On the distribution function of additive functions. *Annals of Mathematics*, 47(1):1–20, 1946.
[2] William J. Firey. Polar means of convex bodies and a dual to the Brunn-Minkowski theorem. *Canadian Journal of Mathematics*, 13:444–453, 1961.
[3] William J. Firey. p-means of convex bodies. *Mathematica Scandinavica*, 10:17–24, 1962.
[4] Richard Gardner, Daniel Hug, and Wolfgang Weil. Operations between sets in geometry. *Journal of the European Mathematical Society*, 15(6):2297–2352, May 2013.
[5] Richard Gardner, Lukas Parapatits, and Franz Schuster. A characterization of Blaschke addition. *preprint*, 2013.
[6] Everett Howe. A new proof of Erdös's theorem on monotone multiplicative functions. *The American Mathematical Monthly*, 93(8):593–595, 1986.
[7] Erwin Lutwak. The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem. *Journal of Differential Geometry*, 38(1):131–150, 1993.
[8] Erwin Lutwak. The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas. *Advances in Mathematics*, 118(2):244–294, 1996.
[9] Vitali Milman and Liran Rotem. α-concave functions and a functional extension of mixed volumes. *Electronic Research Announcements in Mathematical Sciences*, 20:1–11, January 2013.
[10] Vitali Milman and Liran Rotem. Mixed integrals and related inequalities. *Journal of Functional Analysis*, 264(2):570–604, January 2013.
[11] R. Tyrrell Rockafellar. *Convex analysis*. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1970.
[12] Rolf Schneider. *Convex Bodies: The Brunn-Minkowski Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, second edition, 2014.